# On the unipotent $p$-adic Simpson correspondence

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Abstract

The goal of this paper is to show a (derived) $p$-adic Simpson correspondence for (locally) unipotent coefficients on smooth rigid-analytic varieties. Our results depend on a deformation to $\mathbf{B}^+_{\mathrm{dR}}/\xi^2$, and not on a choice of exponential (as required for more general coefficients). Our methods are inherently higher categorical, hinging on the theory of modules over $\mathbf{E}_\infty$-algebras. This paper is a modification of my master thesis at the university of Bonn, defended on March 2023.

## Introduction

The starting point of non-abelian $p$-adic Hodge theory was Deninger-Werner's paper on parallel transport of vector bundles on $p$-adic varieties [10], followed by Falting's paper [11], on which, based on a similar result for complex varieties established by Corlette, Donaldson, Hitchin and Simpson, a corresponence is sketched between

$$\{\text{Higgs Bundles on } X\} \leftrightarrow \{\text{Generalized representations of } \pi_1(X,x)\}$$

over some fixed pointed connected smooth proper curve. A similar correspondence was also conjectured to hold for general proper, smooth varieties over some $p$-adic local field, and a proof is also sketched for "small" objects in a sense made precise in loc. cit. These methods have been worked out on Raynaud's language of formal models, and further research has culminated in the treaty [1] by Abbes, Gross and Tsuji.

Recently, the above correspondence has been studied under the light of perfectoid spaces. In [14], the correspondence has been proven for smooth and proper rigid spaces over some algebraically closed non-archimedian field $C$. Such a decomposition also depends on a choice of deformation of $X$ to $\mathbf{B}^+_{\mathrm{dR}}/\xi^2$ and an exponential map. In [26] a similar correspondence is proven for small coefficients, that does not depend on the choice of such exponential, but only works in good reduction (conjecturally, it should also work if $X$ has semistable reduction).

Our main goal in this paper, is to prove a more special version of the small Simpson correspondence, which holds even without any reduction hypothesis for arbitrary smooth rigid spaces over $C$. Recall that a Higgs bundle (resp. pro-étale vector bundle) is said to be unipotent if it is a successive extension of the unit (cf. Def. 2.3.1). An object is said to be locally unipotent if étale-locally on $X$ it is unipotent.

Theorem (3.4.1). Let $X$ be a smooth rigid-analytic space defined over a closed and complete $p$-adic field $C$ (or mixed characteristic perfectoid with all $p$-power roots of unity) endowed with a (flat) deformation to $\mathbf{B}_{\mathrm{dR}}^{+}/\xi^{2}$. Then there is an equivalence of symmetric monoidal abelian categories

$$\mathsf{Higgs}(X)^{\mathtt{l.uni}} \xrightarrow{\sim} \mathsf{VB}(X_{\mathtt{qpro\acute{e}t}})^{\mathtt{l.uni}};.$$

between pro-étale vector bundles and unipotent Higgs bundles on $X$. A derived analogue of this statement also holds, and in particular this equivalence also preserve the cohomology groups of both sides.

A couple of remarks are in order. Firstly, any rigid-analytic space which is defined over a finite extension of $\mathbf{Q}_p$ will automatically deform canonically since $\overline{K} \subset \mathbf{B}_{\mathrm{dR}}^{+}$ with its direct limit topology. Using spreading out techniques of Conrad and Gabber (see [13, Thm. 7.4.4] for a proof) one also shows that proper rigid-analytic spaces over a closed complete field $C$ admit such deformations (non-canonically).

Secondly, the derived version of such statements is no harder to prove then the non-derived version, provided one has a workable definition of such objects. The right hand side has a site-theoretic definition, but for the left hand side we refer the reader to the main text.

Finally, when proving the correspondence it is enough, by descent, to prove it for unipotent objects and then glue. When $X$ is proper, the unipotent correspondence becomes of a more homotopical nature (as the categories of Higgs bundles and quasi-pro-étale vector bundles are not invariant under, say, $\pi_1$-equivalences). We may then rewrite it in the following form.

Corollary. Let $X$ be a smooth, proper rigid-analytic space defined over a closed and complete $p$-adic field $C$ (or mixed characteristic perfectoid with all $p$-power roots of unity). Fix a geometric point $\bar{x} \to X$ and consider its étale fundamental group $\pi_1(X, \bar{x})$, endowed with its profinite topology. Then there is an equivalence of categories

$$\mathsf{Higgs}(X)^{\mathtt{uni}} \cong \mathsf{Rep}_C(\pi_1(X, \bar{x}))^{\mathtt{uni}}$$

between unipotent continuous representations of $\pi_1$ on $C$-vector spaces and unipotent Higgs bundles on $X$. This equivalence is canonical once fixed a lift of $X$ to $\mathbf{B}_{\mathrm{dR}}^{+}/\xi^{2}$.

Strategy of proof To prove the unipotent correspondence we reinterpret both sides as modules over an approriate $\mathbf{E}_\infty$-algebra, in the sense of [17]. In simple terms, this is an object in a derived category of a (sheaf of) rings which admits an algebra structure whose addition and multiplication laws hold only up to a coherent homotopy.

We now introduce the derived variants of the categories in question. The category of vector bundles on the (quasi-)pro-étale site of $X$ is replaced by the stable infinity category $\mathtt{Perf}(X_{\mathrm{qpro\acute{e}t}})$ of perfect $\widehat{\mathscr{O}}_X$-modules (in the site-theoretic sense); for Higgs bundles the situation is a bit more delicate and we refer to section 2.2 for the definition of $\mathscr{H}$iggs$(X)$.

In [17], one constructs symmetric monoidal stable infinity categories $R$-Mod of modules any $\mathbf{E}_\infty$-algebra $R$. If $R$ is a ordinary commutative ring, then this yields the usual enhancement of the derived category $D(R)$ of $R$. To prove our main theorem, we establish a diagram

$$\begin{array}{ccc} \mathsf{Higgs}(X)^{\mathtt{uni}} & \xrightarrow{\sim} & \mathsf{VB}(X_{\mathtt{qpro\acute{e}t}})^{\mathtt{uni}} \\ \big\downarrow & & \big\downarrow \\ \mathtt{Sym}\,\widetilde{\Omega}^1[-1]\text{-}\mathsf{Mod} & \xrightarrow{\sim} & R\nu_*\widehat{\mathcal{O}}_X\text{-}\mathsf{Mod}; \end{array}$$

where the vertical inclusions (Prop. 1.2.5, Prop. 2.3.9 ) are natural inclusions and the lower horizontal arrow is induced by an isomorphism

$$\Psi\colon \mathtt{Sym}\,\widetilde{\Omega}^1[-1] \xrightarrow{\sim} R\nu_*\widehat{\mathcal{O}}_X,$$

which can be interpreted as a more refined version of the Hodge-Tate decomposition (cf. Sec. 3), and is equivalent to deforming $X$ to $\mathbf{B}_{\mathrm{dR}}^+/\xi^2$. The top horizontal arrow will exist and be an equivalence by purely categorical reasons. The derived statement is proven with the same argument.

The object $\mathtt{Sym}\,\widetilde{\Omega}^1[-1]$, as the name suggests, is the free $\mathbf{E}_\infty$-algebra on the object $\Omega^1[-1]$. It is particularly well behaved in our setting because we are working over $\mathbf{Q}_p$, which is of characteristic zero (Cor. 3.1.3). In particular we need not worry about the distinction between the different versions of $\mathtt{Sym}$.

The object $R\nu_*\widehat{\mathcal{O}}_X$ is the derived pushforward of the unit $\widehat{\mathcal{O}}_X$ of $\mathscr{D}(X_{\mathtt{qpro\acute{e}t}})$ to $\mathscr{D}(X_{\mathtt{\acute{e}t}})$. The importance of the projection $\nu\colon X_{\mathtt{qpro\acute{e}t}} \to X_{\mathtt{\acute{e}t}}$ to $p$-adic Hodge theory was one of the main points of [22] who used the Leray spectral sequenece associated to this morphism to deduce the Hodge-Tate decomposition. The pushforward is an $\mathbf{E}_\infty$-algebras for essentially formal reasons: $R\nu_*$ is lax monoidal because it has a symmetric monoidal left adjoint.

We remark that the isomorphism $\Psi$ can be deduced from any form of the Simpson correspondence that preserves the derived structure (or even the Dolbeault and quasi-pro-étale cohomologies as objects in the derived category of abelian groups) and the symmetric monoidal structure, for essentially formal reasons.

Also as explained in the introduction we are able to extract the argument to its limits using descent and also deduce a correspondence for locally derived unipotent objects in each side. These include all nilpotent Higgs bundles (cf. Section 2.4).

Relation with other works

We also mention that the above theorem has many intersections with the recent developements of the subject. In [26] and [5], we have a correspondence for small objects which is more general and depends also only on a choice of deformation. However, both papers only deal with good reduction case, so our proof is more general. We also point out Tsuji's theorem [1, p. IV.3.4.16] which, as explained in the introduction of the chapter, works for varieties of semistable reduction. However, in all cases above, our proof is fundamentally different, and, in the author's opinion, simpler.

**Notations and conventions** We fix once and for all a prime $p$, a non-archemidean (complete) field $K$ of mixed characteristic and algebraic closure $C$. We use freely the language of adic spaces, and accordingly a rigid variety/space over a non archemidean field $K$ is an adic space $X$ over $\mathrm{Spa}(K, K^\circ)$ which is locally of topologically finite type.

We use blackbold letters to denote our special rings such as the $p$-adic integers $\mathbf{Z}_p$. As usual, $\mathbf{C}_p$ denotes (a choice of) the complete algebraic closure of $\mathbf{Q}_p$.

We also freely use the language of $\infty$-categories in the sense of quasi-categories of Joyal-Lurie; in particular all of our derived categories are therefore considered as stable $\infty$-categories. Hopefully the non-expert can take this infinite-categorical machinery as a blackbox without much effort. We highlight that a theory of higher morphisms is necessary in order to have a good theory of $\mathbf{E}_\infty$-rings.

The free $\mathbf{E}_\infty$ algebra on some complex $\mathscr{M}$ will be denoted by $\mathtt{Sym}\, M$. The free (ordinary) commutative algebra on a module $M$ will be denoted $\mathtt{CSym}\, M$. Since we are in characteristic zero one could identify $\mathtt{Sym}\, M[0] = \mathtt{CSym}\, M$, but we will keep the notational difference for clarity.

Our gradings follow the following convention. Indices on the bottom follows homological conventions, and indices on top follow cohomological ones. They are related via $C_n = C^{-n}$. The cohomology of a complex is denoted by $\mathscr{H}^n$ to possibly distinguish it from its sheaf (hyper)cohomology.

Given a rigid-analytic variety $X$ over $K$, we denote by $\Omega^1_X = \Omega^1_{X/K}$ the sheaf of completed differentials on $X$ (see appendix). We denote the tate twists by $\widetilde{\Omega}^1_X = \Omega^1_X(-1)$ and similarly for $n \in \mathbf{N}$ we have $\widetilde{\Omega}^n_X = \bigwedge^n(\widetilde{\Omega}^1_X)$ and $\widetilde{T}_X = \underline{\mathtt{Hom}}(\widetilde{\Omega}^1_X, \mathscr{O}_X) = T_X(1)$ etc.

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## Contents



# 1 Pro-étale and v-vector bundles

In this section, we review the basic properties of the pro-étale and v-topologies associated to a rigid analytic variety. We define the main objects we are interested in: quasi-pro-étale vector bundles (or, equivalently, v-bundles). We then compare this notion to $C$-local systems and representations of the fundamental group, and show that they agree on unipotent objects when $X$ is proper.

In the appendix we recall notions of diamonds and perfectoid spaces that will be useful in the following.

**§1.1. Pro-étale and v vector bundles; local systems** Any ringed topos comes with a theory of vector bundles and a bounded perfect derived category (see eg. [Stacks, 08G4]). For the quasi-pro-étale/v-topology, this yields one of the sides of the $p$-adic Simpson's correspondence. These objects are related to local systems of $\underline{C}$-modules, where $\underline{C}$ is the sheaf of continuous maps into $C$ as defined in the end of the last section. Furthermore, there is also a relation with continuous $C$ representation of the étale fundamental group. We start by stating a result that guarantees we don't need to care about the difference between the difference between the v and the qproét sites.

Theorem 1.1.1. Let $X$ be an analytic adic space over $\mathrm{Spa}\,\mathbf{Z}_p$, or more generally a diamond. Then pullback along $\lambda\colon X_{\mathrm{v}} \to X_{\mathrm{qpro\acute{e}t}}$ induces equivalences of categories

$$\mathsf{VB}(X_{\mathrm{qpro\acute{e}t}}) \xrightarrow{\sim} \mathsf{VB}(X_{\mathrm{v}}), \quad \mathtt{Perf}(X_{\mathrm{qpro\acute{e}t}}) \xrightarrow{\sim} \mathtt{Perf}(X_{\mathrm{v}}).$$

Remark 1.1.2. Even if one only cares about non-derived objects, the derived result is still important, as it implies that if $M \in \mathscr{D}^b_{\mathrm{perf}}(X_{\mathrm{v}})$, then $R\Gamma_{\mathrm{qpro\acute{e}t}}(X,M) \xrightarrow{\sim} R\Gamma_{\mathrm{v}}(X,\lambda^* M)$.

Proof. Both sides are locally perfectoid, so this theorem reduces to a computation on affinoid perfectoids. For the vector bundle case, this was handled in [25, Lemma 17.1.8]. For perfect objects, the proof is more difficult, and it was done in [3, Thm. 2.1]. □

We now explain the relation between these vector bundles, $C$-local systems and representations of the fundamental group.

From now on we work over a complete algebraically closed non-archimedean perfectoid field $(C,\mathscr{O}_C)$.

Consider a connected rigid-analytic variety $X$ over $C$ and fix a geometric point $\bar{x} \to X$. We define the category (with the usual morphisms)

$$\mathsf{Rep}_C(\pi_1(X,\bar{x})) = \left\{ \begin{array}{c} \text{finite dim. continuous } C\text{-linear} \\ \text{representations of } \pi_1(X,\bar{x}) \end{array} \right\}.$$

This symmetric monoidal abelian category can be identified with the category of local systems on the classifying stack of the fundamental group.

Definition 1.1.3. Let $G$ be a profinite group and $X$ a perfectoid space. A $G$-torsor on $X$ is a perfectoid space $Y/X$ with a $\underline{G}_X$-action which is $\mathtt{pro\acute{e}t}$-locally trivial. We let $BG$ denote the pro-étale stack of $G$-torsors on $\mathsf{Perf}$.

Lemma 1.1.4. Let $X$ be a (locally spatial) diamond. Then the groupoid of morphisms $X \to BG$ is equivalent to the groupoid of $G$-torsors on $X$, that is, the groupoid of (locally spatial) diamonds $Y/X$ with a $\underline{G}_X$ action which is quasi-pro-étale-locally trivial.

Proof. Write $X$ as a quotient by a perfectoid equivalence relation $X^P/R$. A map into $BG$ is the same as a map from the perfectoid $X^P \to BG$ which respects the relation. This is then equivalent to constructing a $G$-torsor $Y/X$. Conversely, any such torsor defines a torsor on $X^P$ respecting $R$, hence defines a map into $BG$. If $X$ is (locally) spatial then so is any $G$-torsor over $X$, since $Y \to X$ is a quasi-pro-étale cover. □

Now we can define a quasi-pro-étale site of $BG$. We say that a morphism of pro-étale stacks $X \to BG$ is quasi-pro-étale if it is locally separated and the pullback to $S \to BG$ is pro-étale for all strictly totally disconnected $S$. The quasi-pro-étale site $BG_{\mathrm{qpro\acute{e}t}}$ is the site of those stacks quasi-pro-étale over $BG$ with v-covers.

As usual, there is a map $\mathrm{pt} \to BG$ which corresponds on $S$ points to the trivial torsor (Here $\mathrm{pt} = \mathrm{Spa}(C,\mathscr{O}_C)$ is the final object). Given a diamond $X$ and a morphism $X \to BG$ classifying a torsor $P$ the diagram

$$\begin{array}{ccc} P & \longrightarrow & \mathtt{pt} \\ \downarrow & & \downarrow \\ X & \longrightarrow & BG \end{array}$$

is cartesian. Hence $\mathtt{pt} \to BG$ is quasi-pro-étale in the sense above and also surjective since $P \to X$ is always surjective [24, Lemma. 10.13].

Proposition 1.1.5. Let $G$ be a profinite group. Then there is a canonical equivalence of symmetric monoidal categories

$$\mathsf{LocSys}(BG_{\mathtt{qpro\acute{e}t}}, C) \xrightarrow{\sim} \mathsf{Rep}_C(G), \quad \mathtt{Perf}(BG_{\mathtt{qpro\acute{e}t}}, C) \xrightarrow{\sim} \mathtt{Perf}(G)$$

given by the pullback to $\mathtt{pt} \to G$.

Proof. Sheaves on any site descend along slice topoi. We conclude that sheaves on $BG_{\mathtt{qpro\acute{e}t}}$ are the same as $G$-equivariant sheaves on $(BG_{\mathtt{qpro\acute{e}t}})_{/\mathtt{pt}}$, that is, condensed sets with a continuous $G$-action. The result follows formally. □

Remark 1.1.6. Seeing $G$ as a group object in the topos of condensed sets (ignoring cardinal issues) then $BG_{\mathtt{qpro\acute{e}t}}$ is identified with the classifying topos of $G$ [SGAIV-IV.2.4] by the proposition above.

Remark 1.1.7. Again, the derived result is important even if one only cares about objects concentrated in degree zero. It implies the cohomological comparison for finite dimensional $C$-representations

$$R\Gamma_{\mathtt{cont}}(G, M) \cong R\Gamma(BG, \underline{M})$$

where the left hand side is defined as the continuous cohomology of $M$ as computed inside the world of condensed sets (which agrees with the classical formula).

Another related object are $C$-local systems. A $C$-local system then is a $\underline{C}$-module which is quasi-pro-étale locally free of finite rank. Here $\underline{C}$ is the quasi-pro-étale sheaf defined before the statement of the primitive comparison theorem (Theorem B.0.13). We claim there are functors

$$\mathsf{Rep}_C(\pi_1(X, \bar{x})) \to \mathsf{LocSys}(C) \to \mathsf{VB}(X_{\mathtt{qpro\acute{e}t}}),$$

relating these objects. To understand these we introduce the pro-étale version of the universal covering space in topology.

Definition 1.1.8 (The universal pro-finite-étale cover). Let $X$ be a connected rigid-analytic variety over $C$, and let $\bar{x} \to X$ be a geometric point. We define the universal pro-finite-étale cover of $X$ to be the limit

$$\widetilde{X} = \lim_{X' \to X} X', \qquad X'(C) \ni \bar{x}' \mapsto \bar{x},$$

of all connected, pointed, finite étale covers $(X', \bar{x}')$ over $(X, \bar{x})$. This limit is taken inside the category of sheaves on Perf, and is a locally spatial diamond.

Almost by definition, the map $\widetilde{X} \to X$ is a quasi-pro-étale (more precisely pro-finite-étale) cover of $X$. It is a torsor under $\pi_1(X,\bar{x})$, every étale cover of $\widetilde{X}$ splits, and any pointed pro-finite-étale cover $(X',\bar{x}') \to (X,\bar{x})$ receives a unique basepoint preserving map $\widetilde{X} \to X'$ (which is automatic pro-finite-étale). Before defining the aforementioned functors, we state some important properties of this covering space.

Proposition 1.1.9. Let $X$ be a connected, qcqs, pointed, rigid-analytic variety over $\mathrm{Spa}\, C$. Then $\widetilde{X}$ satisfies the conditions

$$\mathrm{H}^0(\widetilde{X},C) = C, \quad \mathrm{H}^1(\widetilde{X},C) = 0.$$

If $X$ is also proper, then $\mathrm{H}^0(\widetilde{X},\mathscr{O}_X) = C$.

Proof. This is essentially [15, Prop. 4.9], with some minor adjustements. We note that we have $\mathrm{H}^0(\widetilde{X},\mathscr{O}_C/\varpi^n) = \mathscr{O}_C/\varpi^n$ which follows from [24, Prop. 14.9], and implies $\mathrm{H}^0(\widetilde{X},C) \cong C$. When $X$ is proper, we also get the result on $\widehat{\mathscr{O}}_X$ cohomology via the primitive comparison theorem with the same argument as [15].

For the result on $\mathrm{H}^1$, we first note that

$$\mathrm{H}^1(\widetilde{X},\mathscr{O}_C/\varpi^n) = 0,$$

since any torsor under this sheaf will be trivialized on the inverse limit. This implies that $\mathrm{H}^1(\widetilde{X},C) = 0$ via an $R\lim$ argument. Namely, we have that $\mathscr{O}_C = R\lim_n \mathscr{O}_C/\varpi^n$ by the fact that $X_{\mathtt{qpro\acute{e}t}}$ is replete, so there is a short exact sequence

$$0 \to R^1 \lim_n \mathrm{H}^0(\widetilde{X},\mathscr{O}_C/\varpi^n) \to \mathrm{H}^1(\widetilde{X},\mathscr{O}_C) \to \lim_n \mathrm{H}^1(\widetilde{X},\mathscr{O}_C/\varpi^n) \to 0$$

where the first term vanishes since the projective system in question has surjective transition maps, and the last term vanishes by the argument above. The claim now follows by inverting $p$. □

Now we can use $\widetilde{X}$ to build local systems just as we do in topology. The universal cover $\widetilde{X} \to X$ is classified by a map $l\colon X \to B\pi_1 = B\pi_1(X,\bar{x})$, and since a finite-dimensional continuous $C$-representation of $\pi_1(X,\bar{x})$ can be seen as a $C$ local system on $B\pi_1$, we obtain the first functor via pullback.

Proposition 1.1.10. Let $X$ be a rigid-analytic variety over $\mathrm{Spa}\, C$, pointed and connected. The pullback map via $l\colon X \to B\pi$ defined above determines an exact, symmetric monoidal, fully-faithful functor

$$l^*\colon \mathsf{Rep}_C(\pi_1(X,\bar{x})) \hookrightarrow \mathsf{LocSys}_C(X),$$

whose image consists on all local systems trivialized on a pro-finite-étale cover of $X$. Furthermore for a representation $V$ we have

$$\mathrm{H}^0_{\mathtt{cont}}(\pi_1,V) \xrightarrow{\sim} \mathrm{H}^0_{\mathtt{qpro\acute{e}t}}(X,l^*V), \quad \mathrm{H}^1_{\mathtt{cont}}(\pi_1,V) \xrightarrow{\sim} \mathrm{H}^1_{\mathtt{qpro\acute{e}t}}(X,l^*V).$$

Proof. It is clear from the universal property of $\widetilde{X}$ that a local system is trivialized on a pro-finite-étale cover of $X$ if and only if it is trivial on $\widetilde{X}$. Therefore we pass to the subcategory

$$\mathsf{lLocSys}_C(X) \subset \mathsf{LocSys}_C(X)$$

of such pro-finite-étale local systems and we show that pullback induces an equivalence. Note that the following diagram

$$\begin{array}{ccc} \widetilde{X} & \longrightarrow & \operatorname{Spa} C \\ \downarrow & & \downarrow \\ X & \longrightarrow & B\pi_1 \end{array}$$

is cartesian, which implies that the essential image of the pullback does indeed lie in $\mathsf{lLocSys}_C(X)$ since all local systems on $\operatorname{Spa} C$ are trivial.

Now the inverse of the pullback is given by the functor

$$L \mapsto \Gamma(L, \widetilde{X})$$

which is a finite-dimensional $C$ vector space (by proposition 1.1.9) with a continuous $\pi_1(X, \bar{x})$-action induced from the action on $\widetilde{X}$.

The results about cohomology follow straight from the Čech-to-sheaf cohmology spectral sequence and the computation $\mathrm{H}^1(\widetilde{X}, C) = 0$ on proposition 1.1.9 (since the Čech cohomology of $\widetilde{X}/X$ computes the continuous cohomology of those local systems which become trivial on it). □

Remark 1.1.11. One can show that $\mathsf{lLocSys}_C(X)$ are the same as local systems $L$ which admit a $\mathscr{O}_C$-lattice, that is, a sub-$\mathscr{O}_C$-module $\mathscr{L} \subset L$ such that $\mathscr{L}$ is a $\mathscr{O}_C$-local system and $L = \mathscr{L}[1/p]$.

There is also a map $\mathsf{LocSys}_C(X) \to \mathsf{VB}(X_{\mathtt{qproét}})$ which is much simpler to describe. It is simply given by base change:

$$L \mapsto L \otimes_C \widehat{\mathscr{O}}_X \in \mathsf{VB}(X_{\mathtt{qproét}}).$$

This map is not as well behaved as the first one. There is no hope of this functor being fully-faithful unless $X$ is proper since

$$C \cong \mathrm{Hom}(C, C) \to \mathrm{Hom}(\widehat{\mathscr{O}}_X, \widehat{\mathscr{O}}_X) \cong \Gamma(X, \widehat{\mathscr{O}}_X)$$

is not necessarily an isomorphism. However, even if $X$ is proper we can only guarantee that

$$\mathrm{Hom}(L_1, L_2) \xrightarrow{\sim} \mathrm{Hom}(L_1 \otimes \widehat{\mathscr{O}}_X, L_2 \otimes \widehat{\mathscr{O}}_X)$$

is an isomorphism when $L_1$ and $L_2$ are trivialized by some quasi-pro-étale cover $Y \to X$ with $\mathrm{H}^0(Y, \widehat{\mathscr{O}}_X) = C$. Taking into consideration 1.1.9 we obtain a fully faithful functor $\mathsf{lLocSys}_C(X) \hookrightarrow \mathsf{VB}(X_{\mathtt{qproét}})$. The image of such functor is clear: it consists on those vector bundles which are trivial over a pro-finite-étale cover of $X$. Let $\mathsf{VB}(X)_{\mathtt{profét}}$ denote the category of such bundles [1].

[1] Not to be confused with $\mathsf{VB}(X_{\mathtt{profét}})$, which are vector bundles in the pro-finite-étale topology. These are identified canonically with $\mathsf{Rep}_C(\pi_1(X, \bar{x}))$, and therefore only agrees with $\mathsf{VB}(X)_{\mathtt{profét}}$ when $X$ is proper by this theorem.

Theorem 1.1.12 ([15, Thm. 5.2]). Let $X$ be a proper, rigid analytic space over $C$, with a fixed geometric point $\bar{x} \hookrightarrow X$. The functors defined above define exact, symmetric monoidal equivalences

$$\mathsf{Rep}_C(\pi_1(X,\bar{x})) \xrightarrow{\sim} \mathsf{lLocSys}_C(X) \xrightarrow{\sim} \mathsf{VB}(X)_{\mathtt{prof\acute{e}t}}.$$

Proof. Follows from the above discussion. □

Remark 1.1.13. The equivalence $\mathsf{lLocSys}_C(X) \xrightarrow{\sim} \mathsf{VB}(X)_{\mathtt{prof\acute{e}t}}$, in contrast with the first one, preserves all cohomology groups, and hence can be enriched to an equivalence

$$\mathtt{Perf}(X,C)_{\mathtt{prof\acute{e}t}} \xrightarrow{\sim} \mathtt{Perf}(X,\widehat{\mathscr{O}}_X)_{\mathtt{prof\acute{e}t}},$$

where the subscript $\mathtt{prof\acute{e}t}$ means we are considering objects trivialized over a pro-finite-étale cover (equivalently $\widetilde{X}$). This follows from the full primitive comparison theorem, which is analogous to Theorem B.0.13, but works for arbitrary local systems. As we will focus on unipotent local systems later on, we will simply deduce the cohomological comparison from the case of the unit, to emphasize this unipotent technique.

Remark 1.1.14. The above theorem is a non-archimedian analogue of a well known phenomenon. For complex manifolds, the analogous functors (where $\mathbf{C}$ is denotes the complex numbers)

$$\mathsf{Rep}_{\mathbf{C}}(\pi_1(X,x)) \xrightarrow{\sim} \mathsf{LocSys}_{\mathbf{C}}(X) \xrightarrow{\sim} \mathsf{VB}^{\nabla}(X_{\mathrm{an}})$$

identify all local systems as coming from a representation (a very weak version of Riemann-Hilbert). Furthermore, the category of quasi-pro-étale bundles becomes the category of analytic bundles endowed with a flat connection.

In our case, we observe that even if one wanted to define a flat connection on a quasi-pro-étale vector bundle, this could not be the naive definition, as we know that this topology is locally perfectoid, and those spaces have, in some sense, no differentials.

In terms of the proof given above, this difference is related to the fact that the (topological) universal cover of $X$ is almost never compact (so it has too many global sections).

§1.2. Unipotent bundles and $Rv_*\widehat{\mathscr{O}}_X$-modules. We are interested in unipotent objects in these categories. If $\mathsf{C}$ is a symmetric monoidal abelian category, we denote by $\mathsf{C}^{\mathtt{uni}}$ the full subcategory of $\mathsf{C}$ generated by unipotent objects, that is, successive extensions of the unit.

There is also a derived version of unipotence. Let $\mathscr{D}$ be a symmetric monoidal stable infinity category. We denote by $\mathscr{D}^{\mathtt{uni}} \subset \mathscr{D}$ the smallest stable subcategory of $\mathscr{D}$ that contains the unit. An object of $\mathscr{D}^{\mathtt{uni}}$ is called derived unipotent.

In general, even if $\mathsf{C} \subset \mathscr{D}$ is the heart of some $\mathtt{t}$-structure, we cannot guarantee that the notions of derived unipotence and unipotence agree. For example, if $\mathsf{C}$ is the category of finitely generated $R$ modules, for $R$ a PID, and $\mathscr{D} = \mathtt{Perf}(R)$, the only unipotent $R$-modules are the trivial ones ($R$ is projective) but all finite modules are derived unipotent since they can be written as a cone $M \cong \mathrm{Cof}(R^m \to R^n)$. However, the converse always holds, as we show here below.

For concreteness, we also note we have derived variants of all objects we are considering. Instead of $\mathsf{Rep}_X(\pi_1)$, $\mathsf{LocSys}_C(X)$ and $\mathsf{VB}(X_{\mathtt{qpro\acute{e}t}})$ we can consider

$$\mathtt{Perf}(B\pi_1,C),\ \mathtt{Perf}(X,C),\ \mathtt{Perf}(X,\widehat{\mathscr{O}}_X)$$

respectively (all underlying topologies are taken to be quasi-pro-étale).

Lemma 1.2.1. Let $X$ be a rigid-analytic variety over $C$. Let $\mathsf{C}$ be either $\mathsf{Rep}_C(\pi_1(X,\bar{x}))$, $\mathsf{LocSys}_C(X)$ or $\mathsf{VB}(X_{\mathtt{qpro\acute{e}t}})$, and let $\mathscr{D}$ be its derived variant. Then

$$\mathsf{C}^{\mathtt{uni}} \subset \mathscr{D}^{\mathtt{uni}} \cap \mathsf{C},$$

that is, every unipotent object is derived unipotent.

Proof. To see that every unipotent object is derived unipotent, we just note that if $0 \to E' \to E \to E'' \to 0$ is an extension, then $E = \mathrm{Fib}(E'' \to E[1])$ in the derived variant, so $E$ is derived unipotent by induction on the rank. □

Theorem 1.2.2. Let $X$ be a proper and connected rigid-analytic variety over $C$. The functors above induce equivalences on unipotent objects

$$\mathsf{Rep}_C(\pi_1(X,\bar{x}))^{\mathtt{uni}} \xrightarrow{\sim} \mathsf{LocSys}_C(X_{\mathtt{qpro\acute{e}t}})^{\mathtt{uni}} \xrightarrow{\sim} \mathsf{VB}(X_{\mathtt{pro\acute{e}t}})^{\mathtt{uni}}.$$

For derived unipotent objects, we also have $\mathtt{Perf}(X,C)^{\mathtt{uni}} \xrightarrow{\sim} \mathtt{Perf}(X,\widehat{\mathscr{O}}_X)^{\mathtt{uni}}$.

Proof. We note that all functors are symmetric monoidal, since they are induced by pullback maps of ringed topoi. We start by proving that the second arrow is fully faithful (classical or derived). Since all objects are dualizable in these categories, we have that

$$\mathrm{Hom}(X,Y) = \mathrm{Hom}(\mathbf{1}, X^\vee \otimes Y),$$

so in particular, it is enough to consider maps $\mathbf{1} \to L$, for a local system $L$. Therefore we reduce the above question to an extension of the primitive comparison theorem (Thm. B.0.13)

$$R\Gamma(X,L) \xrightarrow{\sim} R\Gamma(X, L \otimes \widehat{\mathscr{O}}_X)$$

to all unipotent coefficients.

For the classical (non-derived) case, this follows by the base case and an induction on the rank. Namely, we can find an extension $0 \to L' \to L \to C \to 0$ which induces a diagram

$$\begin{array}{ccccc}
R\Gamma(L') & \longrightarrow & R\Gamma(L) & \longrightarrow & R\Gamma(C) \\
\downarrow{\scriptstyle\sim} & & \downarrow & & \downarrow{\scriptstyle\sim} \\
R\Gamma(L' \otimes \widehat{\mathscr{O}}_X) & \longrightarrow & R\Gamma(L \otimes \widehat{\mathscr{O}}_X) & \longrightarrow & R\Gamma(\widehat{\mathscr{O}}_X)
\end{array}$$

which implies that $R\Gamma(L) \xrightarrow{\sim} R\Gamma(L \otimes \widehat{\mathscr{O}}_X)$ as required. For the derived case, we just consider the subcategory of all objects $\mathscr{L}$ such that the arrow above is an isomorphism. We then note that $C$ is in such category, and its closed under shifts and fibers (by a variation of the argument above), and hence it is $\mathtt{Perf}(X,C)^{\mathtt{uni}}$.

For essential surjectivity in the classical case, it is enough to see that if $V$ is a unipotent representations of the fundamental group, then any extension of $V$ by $\widehat{\mathscr{O}}_X$, as a quasi-pro-étale vector bundle, comes from another representation. This translates to the question of whether the maps

$$\mathrm{H}^1_{\mathtt{cont}}(\pi_1(X,\bar{x}),V) \xrightarrow{\sim} \mathrm{H}^1(X,L) \xrightarrow{\sim} \mathrm{H}^1(X,L\otimes\widehat{\mathscr{O}}_X),$$

are isomorphisms, which we know from Proposition 1.1.10 and the discussion above.

The only thing left to argue is that the functor $\mathtt{Perf}(X,C)^{\mathtt{uni}} \hookrightarrow \mathtt{Perf}(X,\widehat{\mathscr{O}}_X)^{\mathtt{uni}}$ is essentially surjective. But that is easy, as it is exact and hence the image is stable and contains $\widehat{\mathscr{O}}_X$. □

Remark 1.2.3. The functor

$$\mathtt{Perf}(\pi_1,C) \to \mathtt{Perf}(X,C)$$

is not fully faithful, even when restricted to unipotent objects. This makes sense, because we do not expect in general for étale cohomology to be computed as group cohomology. When this happens, we could say that $X$ is a $K(\pi,1)$ for $p$-adic coefficients.

Corollary 1.2.4. Let $X$, $Y$ be smooth, proper, connected, pointed rigid-analytic varieties over $C$ and suppose that a pointed map $f: X \to Y$ induces an equivalence

$$\pi_1(X,\bar{x}) \xrightarrow{\sim} \pi_1(Y,f\bar{x}).$$

Then it also induces via pullback symmetric monoidal equivalences $\mathsf{VB}(Y_{\mathtt{qpro\acute{e}t}})^{\mathtt{uni}} \xrightarrow{\sim} \mathsf{VB}(X_{\mathtt{qpro\acute{e}t}})^{\mathtt{uni}}$, and $\mathsf{LocSys}(Y)^{\mathtt{uni}} \xrightarrow{\sim} \mathsf{LocSys}_C(X)^{\mathtt{uni}}$.

We now finish this section by relating unipotent quasi-pro-étale vector bundles on $X$ to modules over its derived endomorphism algebra. Here we recall the existence of a lax functor

$$R\nu_*: \mathtt{Perf}(X_{\mathtt{qpro\acute{e}t}},\widehat{\mathscr{O}}_X) \to \mathscr{D}(X_{\mathtt{\acute{e}t}}),$$

which defines for us a sheaf of $\mathbf{E}_\infty$-algebras $R\nu_*\widehat{\mathscr{O}}_X$ on $X_{\mathtt{\acute{e}t}}$.

Proposition 1.2.5. Let $X$ be a rigid-analytic variety. The above functor defines a symmetric monoidal equivalence

$$\mathtt{Perf}(X,\widehat{\mathscr{O}}_X)^{\mathtt{uni}} \xrightarrow{\sim} R\nu_*\widehat{\mathscr{O}}_X\text{-}\mathsf{Mod}^{\mathtt{uni}},$$

which identifies $\mathsf{VB}(X_{\mathtt{qpro\acute{e}t}})^{\mathtt{uni}}$ as the smallest subcategory of the right hand side which contains the unit and is closed under extensions (in the sense of fiber sequences).

Proof. The proof is similar to the theorem above, but now the work lies in showing that the lax structure maps

$$R\nu_*\mathscr{E} \otimes_{R\nu_*\widehat{\mathscr{O}}_X} R\nu_*\mathscr{E}' \xrightarrow{\sim} R\nu_*(\mathscr{E}\otimes\mathscr{E}')$$

are equivalences. We hence fix $\mathscr{E}$ and consider the full subcategory of all $\mathscr{V}$ such that the lax morphism above is an equivalence. Then this category contains the unit and is stable since it is closed under shifts and fiber sequences: if $\mathscr{V}' \to \mathscr{V} \to \mathscr{V}''$ is a fiber sequence then

$$\begin{array}{ccccc} R\nu_*\mathscr{V}' \otimes_{R\nu_*\widehat{\mathscr{O}}_X} R\nu_*\mathscr{E} & \longrightarrow & R\nu_*\mathscr{V} \otimes_{R\nu_*\widehat{\mathscr{O}}_X} R\nu_*\mathscr{E} & \longrightarrow & R\nu_*\mathscr{V}'' \otimes_{R\nu_*\widehat{\mathscr{O}}_X} R\nu_*\mathscr{E} \\ \downarrow & & \downarrow & & \downarrow \\ R\nu_*(\mathscr{V}'\otimes\mathscr{E}) & \longrightarrow & R\nu_*(\mathscr{V}\otimes\mathscr{E}) & \longrightarrow & R\nu_*(\mathscr{V}''\otimes\mathscr{E}) \end{array}$$

commutes and both rows are fiber sequences, hence if two of the vertical arrows are isomorphisms, so is the third.

Now the proof follows the same arguments as before. Namely we use that every object is dualizable to reduce fully faithfulness to the claim that

$$R\Gamma_{\text{ét}} \circ R\nu_* \cong R\Gamma,$$

where the last global sections are taken in the quasi-pro-étale site, but this is clear. Essential surjectivity follows by stability of the image.

The second claim now follows suit, since any extension in $\mathtt{Perf}(X, \widehat{\mathcal{O}}_X)$ of objects in $\mathsf{VB}(X_{\text{qproét}}, \widehat{\mathcal{O}}_X)$ will automatically come from a vector bundle (take long exact sequence in cohomology, note that the extension is concentrated in degree zero and that the category of unipotent vector bundles is closed under extensions). We also note that this is a monoidal subcategory, since vector bundles are flat. □

## 2 Higgs bundles

We start by finally defining the other side of the correspondence: Higgs bundles. These objects were first defined on curves by Hitchin, and generalized to higher dimensional varieties by Simpson.

§2.1. Higgs bundles

In this section we define Higgs bundles from a "hands-on" perspective, as a rigid tensor category. We will later see that Higgs bundles also admit a derived version, as a symmetric monoidal stable infinity-category.

We recall that we denote by $\widetilde{\Omega}^1 = \Omega^1(-1)$ the sheaf of twisted differentials. Similarly $\widetilde{\Omega}^n = (\widetilde{\Omega}^1)^{\otimes n} = \Omega^n(-n)$ for $n \in \mathbf{Z}$. (The inverse is the dual.) Similarly, $\widetilde{T}_X = T_X(1) = \widetilde{\Omega}^{-1}$.

Definition 2.1.1. Let $X$ be a smooth rigid-analytic variety over $C$, $E$ a vector bundle on $X_{\text{ét}}$, and $\widetilde{\Omega}^1$ the bundle of twisted differentials. A Higgs field on $E$ is a global section $\theta \in \Gamma(X, \mathrm{End}(E) \otimes \widetilde{\Omega}^1)$ subject to the condition that

$$\theta \wedge \theta = 0 \quad \text{in } \widetilde{\Omega}^2 \otimes \mathrm{End}(E)$$

A Higgs bundle is a vector bundle $E$ on $X_{\text{ét}}$ endowed with a Higgs field $\theta$.

A morphism of Higgs bundle is a morphism of the underlying $\mathcal{O}_X$-modules commuting with the Higgs field. We denote the category of Higgs bundles by $\mathsf{Higgs}(X)$.

Remark 2.1.2. Locally, when $E \cong \mathcal{O}_X^n$, a section $\theta \in \Gamma(X, \mathrm{End}(E) \otimes \widetilde{\Omega}^1)$ can be seen as a choice of differential forms $\omega_1, \ldots, \omega_n \in \Gamma(X, (\widetilde{\Omega}^1)^n)$, and the condition $\theta \wedge \theta = 0$ translates to

$$\omega_i \wedge \omega_j = 0,$$

ie, the sections commute. Since $\widetilde{\Omega}^1$ is dualizable, we can also write the Higgs field as

$$\theta\colon \widetilde{T}_X \to \mathrm{End}(E), \qquad \text{or even } \theta\colon \widetilde{T}_X \otimes E \to E,$$

where $\widetilde{T}_X$ is the twisted tangent bundle. The condition above then translates to commutativity of the image, that is, $[\theta,\theta] = 0$; or equivalently that it extends to a morphism from the free commutative algebra sheaf $\mathtt{CSym}\,\widetilde{T}_X \to \mathrm{End}(E)$.

We conclude that Higgs bundles are the same thing as modules over $\mathtt{CSym}\,\widetilde{T}_X$ whose underlying sheaf is a vector bundle. We will return to this point later when introducing derived Higgs bundles.

Given a Higgs bundle $(E,\theta)$, we can deduce a morphism

$$\theta_2 \colon E \otimes \widetilde{\Omega}^1 \xrightarrow{\theta\otimes 1} E \otimes \widetilde{\Omega}^1 \otimes \widetilde{\Omega}^1 \xrightarrow{\wedge} E \otimes \widetilde{\Omega}^2.$$

using the wedge product on differential forms. We observe that the condition $\theta_2 \circ \theta = 0$ is the condition for a Higgs field. This process naturally extends to higher forms.

Definition 2.1.3. Let $(E,\theta)$ be a Higgs bundle. We define the Higgs complex (or Dolbeaut complex) of $E$ to be

$$\mathscr{A}(X,E) = \left[\mathscr{E} \xrightarrow{\theta} E \otimes \widetilde{\Omega}^1 \xrightarrow{\theta_2} E \otimes \widetilde{\Omega}^2 \xrightarrow{\theta_3} \dots\right] \in \mathscr{D}(X_{\text{ét}}).$$

The cohomology of this complex $R\Gamma_{\mathtt{Dol}}(E) = R\Gamma_{\text{ét}}(\mathscr{A}(X,E))$ is called the Dolbeaut cohomology of $E$.

Remark 2.1.4. A remark on the nomenclature: if $\mathscr{O}_X$ is equipped with the 0 Higgs field, then its Dolbeaut complex splits

$$\mathscr{A}(X,\mathscr{O}_X) = \bigoplus_i \widetilde{\Omega}^i[-i]$$

and hence its (hyper)cohomology agrees with the usual definition of Dolbeaut cohomology of $X$. We will later see that this remark is a main ingredient in proving the unipotent $p$-adic Simpson correspondence.

Proposition 2.1.5. The category $\mathsf{Higgs}(X)$ admits a canonical closed symmetric monoidal structure making the forgetful functor $\mathsf{Higgs}(X) \to \mathsf{VB}(X_{\text{ét}})$ strong monoidal. This category is then rigid: every Higgs bundle is dualizable.

Proof/Def. If $\mathscr{F}$ and $\mathscr{G}$ are Higgs bundles with fields both denoted $\theta$, then the tensor product $\mathscr{F} \otimes \mathscr{G}$ becomes a Higgs bundle using the Leibniz rule

$$\theta(v \otimes w) = \theta(v) \otimes w + v \otimes \theta(w),$$

noting that it squares to zero. The unit for this monoidal structure is just the vector bundle $\mathscr{O}_X$, with the zero Higgs field.

Similarly, this monoidal structure is closed, with internal hom $\underline{\mathtt{Hom}}(\mathscr{F},\mathscr{G})$ and underlying Higgs field $\theta(f) = \theta f - f\theta$. Note that

$$\mathrm{Hom}_{\mathsf{Higgs}}(\mathscr{O}_X,\mathscr{F}) = \ker(\Gamma(X,\mathscr{F}) \xrightarrow{\theta} \Gamma(X,\mathscr{F} \otimes \Omega^1)) = \mathrm{H}^0_{\mathtt{Dol}}(\underline{\mathtt{Hom}}(X,\mathscr{F})).$$

We also note that as usual for closed symmetric monoidal categories, the dual of some Higgs bundle $E$ has to be $E^\vee = \underline{\mathtt{Hom}}(E,\mathscr{O}_X)$. That every Higgs bundle is dualizable follows from the above construction and the rigidity of $\mathsf{VB}(X_{\text{ét}})$. □

§2.2. Derived Higgs bundles We also discuss the notion of a derived Higgs bundle (called Higgs perfect complex in [4]). Informally, we can think of these objects as perfect complexes on $T_X^*$ which such that the pushfoward $\pi_*$ is perfect on $X$. All sheaves and perfect objects are taken with regards to the analytic topology for convenience[2].

In this section we denote by $H_X = \mathtt{CSym}\,\widetilde{T}_X$ denotes the classical free commutative "symmetric" algebra on the twisted tangent bundle.Locally, when $X = \mathrm{Spa}(A, A^+)$, $H_X$ corresponds to $H_A$, the $A$-algebra

$$H_A = \mathtt{CSym}_A\,\widetilde{\Omega}_A^\vee = \bigoplus_{n=0}^{\infty}(\widetilde{\Omega}_A^{\vee\otimes n})_{\Sigma_n}$$

Locally, when $X$ admits an étale map to a torus $\mathrm{Spa}(C\langle T_1 \dots T_n\rangle)$, $H_A$ is further isomorphic to $H_A = A[T_1, \dots, T_n]$.

We can see $H_X$ as an analytic sheaf of rings in $X$. Sheaves of $H_X$-modules can be seen as usual as $H_X$-modules in $\mathscr{D}(X) = \mathscr{D}(\mathscr{O}_X)$ via its $\mathscr{O}_X$-algebra structure. We have a forgetful functor

$$\pi_* : \mathscr{D}(H_X) = H_X\text{-}\mathsf{Mod}(\mathscr{D}(X)) \to \mathscr{D}(X)$$

which we denote by $\pi_*$ for a more geometric intuition. Using the theory of quasi-coherent sheaves on rigid-analytic spaces, we can also see $H_X$ as an algebra inside of $\mathscr{D}_{\mathrm{qc}}(X)$, and $\mathscr{D}(H_X) \subset H_X\text{-}\mathsf{Mod}(\mathscr{D}_{\mathrm{qc}}(X))$.

Definition 2.2.1. Let $X$ be a smooth rigid-analytic variety over $C$. The category of derived Higgs bundles is defined to be the full subcategory

$$\mathscr{H}\mathrm{iggs}(X) = \mathtt{Perf}(H_X) \times_{\mathscr{D}(\mathscr{O}_X)} \mathtt{Perf}(\mathscr{O}_X) \subset \mathtt{Perf}(H_X)$$

of $\mathtt{Perf}(H_X)$ consisting of all objects which are already perfect over $\mathscr{O}_X$, meaning all objects $\mathscr{E}$ such that $\pi_*\mathscr{E}$ lies in $\mathtt{Perf}(X) \subset \mathscr{D}(X)$. An object in $\mathscr{H}\mathrm{iggs}(X)$ is called a derived Higgs bundle.

Remark 2.2.2. Since the forgetful functor $\pi_* : \mathtt{Perf}(H_X) \to \mathscr{D}(X)$ is an exact functor between stable $\infty$-categories, $\mathscr{H}\mathrm{iggs}(X)$ is also stable. That is, $\mathscr{H}\mathrm{iggs}(X)$ is stable under (co)fiber and shifts as a subcategory of $\mathtt{Perf}(H_X)$.

In other to be able to work with and justify this definition, we need to make sure that Higgs bundles are derived Higgs bundles. In order to do so, we recall the descent result for perfect objects proven in [2, Thm. 1.4]. More precisely, the functor

$$U \mapsto \mathtt{Perf}(A),$$

where $U = \mathrm{Spa}(A, A^+)$ is an affinoid of $X$, is an analytic sheaf of $\infty$-categories. In particular, we conclude that if $X = \mathrm{Spa}(A, A^+)$ is affinoid then the canonical functor

$$\mathtt{Perf}(A) \xrightarrow{\sim} \mathtt{Perf}(X)$$

is an equivalence.

[2]But as we will see in Proposition 2.2.6, one could in principle define derived Higgs bundles for the étale topology instead.

Proposition 2.2.3. Let $X$ be a smooth rigid-analytic variety over $C$. Then there is a fully faithful functor

$$\mathsf{Higgs}(X) \hookrightarrow \mathscr{H}\mathrm{iggs}(X)$$

which agrees with $\mathsf{VB}(X) \hookrightarrow \mathsf{Perf}(X)$ on the underlying $\mathscr{O}_X$-modules.

Proof. The data of a Higgs bundle is, as pointed out in the remark after Def. 2.1.1, just an $H_X$-module $E$ which is a vector bundle over $\mathscr{O}_X$. Hence we need to check that the $H_X$-module structure induced from $E$ is already perfect over $H_X$. (We also remark that there is no reason for $E$ to have tor amplitude 0 as an $H_X$-module.)

Now, this is a local problem so we may assume $X$ to be affinoid, of the form $X = \mathrm{Spa}(A, A^+)$ and $\widetilde{\Omega}^1_X \cong \mathscr{O}_X[T_1, \ldots, T_n]$ to be polynomial. Then $\mathsf{Higgs}(X)$ can be identified with the category of pairs $(E, \theta)$ with $E \in \mathsf{VB}(A)$ and $A$ and a Higgs field $\theta \colon E \to \Omega^1_A$. This is then a problem on the underlying rings, and we reduce to the following lemma, which is a variation of [4, Lemma 6.17] that works in our situation. □

Lemma 2.2.4. Let $X$ be a rigid-analytic affinoid variety over $C$, and $\mathscr{A}$ a quasi-coherent $\mathscr{O}_X$-algebra. Let also $H = \mathscr{A}[T]$ is the free quasi-coherent $\mathscr{O}_X$-algebra on an element $T$. Let $\mathscr{D}_{\mathrm{qc}}(H) = H\text{-}\mathsf{Mod}(\mathscr{D}_{\mathrm{qc}}(X))$ and $\pi_* : \mathscr{D}_{\mathrm{qc}}(H) \to \mathscr{D}_{\mathrm{qc}}(X)$ be the forgetful functor. It has a left adjoint $\pi^* : \mathscr{D}_{\mathrm{qc}}(X) \to \mathscr{D}_{\mathrm{qc}}(H)$ given by tensoring with $H$.

Then every $\mathscr{E} \in \mathscr{D}_{\mathrm{qc}}(H)$ gives us a fiber sequence

$$\pi_*\mathscr{E} \otimes^L_{\mathscr{O}_X} H \xrightarrow{T\otimes 1 - 1\otimes T} \pi_*\mathscr{E} \otimes^L_{\mathscr{O}_X} H \longrightarrow \mathscr{E}$$

where the second map is the counit.

In particular, by induction and the fact that pulling back preserves perfectness, if $\mathscr{E}$ is perfect over $\mathscr{O}_X$, then it is perfect over $\mathscr{O}_X[T_1, \ldots, T_n]$ for all $n$.

Proof. Let $X = \mathrm{Spa}(A, A^+)$ and $\mathscr{M}$ the measures of the underlying analytic ring. Since $\mathscr{D}_{\mathrm{qc}}(X)$ is generated by colimits and shifts by $\mathscr{M}[S]$, for $S$ extremelly disconnected. Therefore The category $\mathscr{D}_{\mathrm{qc}}(H) = H\text{-}\mathsf{Mod}(\mathscr{D}_{\mathrm{qc}}(X))$ is generated (under colimits and shifts) by $\pi^*\mathscr{M}[S] = \mathscr{M}[S] \otimes^L_A H$, so we may assume that $\mathscr{E}$ is of this form. But the sequence for $\pi^*\mathscr{M}[S]$ is just the sequence for $\pi^*\mathscr{M}[*] = H$ tensored over $H$ with the $\pi^* M[S]$, so we can further reduce to the case of $\mathscr{E} = H$. Now it is a mere check, as in [4, Lemma 6.17]. □

Corollary 2.2.5. Let $X = \mathrm{Spa}(A, A^+)$ be a smooth affinoid rigid-analytic variety over $C$ ith trivial cotangent bundle (and hence $H_X = \mathscr{O}_X[T_1, \ldots, T_n]$). Then the category of derived Higgs bundles can be identified with

$$\mathscr{H}\mathrm{iggs}(X) = \mathtt{Perf}(H_A) \times_{\mathscr{D}(A)} \mathtt{Perf}(A).$$

Proof. We have an inclusion $\mathtt{Perf}(H_A) \subset \mathtt{Perf}(H_X)$. Let $\mathscr{E}$ be a derived Higgs bundle on $X$, that is, suppose that $\pi_*\mathscr{E}$ is a perfect $\mathscr{O}_X$-module. We want to show that in fact $\mathscr{E}$ is in the image $\mathtt{Perf}(H_A)$. By descent for perfect objects, we have that $\mathtt{Perf}(X) = \mathtt{Perf}(A)$, and hence, by induction and the lemma above, we have our result. □

Now, our main theorem is a local unipotent Simpson correspondence. We can extend to a stronger global correspondence using descent for Higgs bundles.

Proposition 2.2.6. The association $X \mapsto \mathscr{H}\mathrm{iggs}(X)$ can be enhanced to a functor from $X_{\text{ét}}^{\mathrm{op}}$ to $\infty$-categories. The associated prestack (passing to the core $\mathscr{H}\mathrm{iggs}(X)^{\cong}$) is a stack, that is, we have étale descent for Higgs bundles.

Proof. For the functoriality, it is enough to show that if $f: Y \to X$ is an étale morphism of smooth rigid varieties then the morphism

$$f^*: \mathtt{Perf}(H_X) \to \mathtt{Perf}(H_Y)$$

induced by the isomorphism $f^*H_X \xrightarrow{\sim} H_Y$ of $\mathscr{O}_Y$-algebras preserves the categories of derived Higgs bundles. But this is clear since the underlying $\mathscr{O}_Y$-module of $f^*\mathscr{E}$ is the pullback functor $\mathtt{Perf}(Y) \to \mathtt{Perf}(X)$.

Analytic descent for $\mathscr{H}\mathrm{iggs}(X)$ is clear by definition, since perfect modules descend for every topos. We can then reduce the general étale case to $X = \mathrm{Spa}(A, A^+)$ and $Y = \mathrm{Spa}(B, B^+)$ with $B/A$ finite étale and such that the cotangent bundles are trivial. By the Corollary above, we reduce to usual étale descent for perfect complexes, since $H_A \to H_A \otimes_A B \cong H_B$ is étale. □

Remark 2.2.7. In usual algebraic geometry, the category of (derived) Higgs bundles can be identified as a subcategory of sheaves on the cotangent complex. In rigid geometry this is more subtle classically, as the notion of affinoid morphisms is not as simple as its discrete counterpart.

However, we observe that the category $\mathscr{D}_{\mathrm{qc}}(H_X)$, locally, can be identified with a category of modules over an analytic ring in the sense of Clausen-Scholze. Indeed, its the modules over the completion of $H_A$ for the canonical analytic ring structure in $H_A$ coming from the map $A \to H_A$ and the analytic ring structure of $A$. This should be identified with the category of quasi-coherent sheaves on the (geometric) cotangent bundle of X, and $\pi_*$, $\pi^*$ and $s_*$ should be identified with their geometric counterparts.

### 2.2.1 The symmetric monoidal structure, and the cohomology of Higgs bundles

The sheaf $H_X$, seen as as object in $\mathtt{Perf}(H_X)$, is not a derived Higgs bundle, and therefore $\mathscr{H}\mathrm{iggs}(X)$ does not inherit the canonical symmetric monoidal structure from $\mathtt{Perf}(H_X)$. On the other hand, we have a Hopf algebra structure on $H_X$, and therefore $\mathtt{Perf}(H_X)$ admits another symmetric monoidal structure making the forgetful functor $\pi_*$ symmetric monoidal. Concretely, this means that $H_X$ is also a co-algebra, with maps

$$H_X \to H_X \otimes H_X, \quad H_X \to \mathscr{O}_X$$

which in local coordinates (fixing an étale map to a torus with coordinates $T_i$) are of the form $T_i \mapsto T_i \otimes 1 + 1 \otimes T_i$ and $T_i \mapsto 1$. This symmetric monoidal structure then comes from the derived category of $H_X$-modules, as the left derived functor of the usual closed symmetric monoidal structure on $H_X$-modules.

One also has a symmetric monoidal inclusion

$$s_*: \mathtt{Perf}(X) \hookrightarrow \mathtt{Perf}(H_X)$$

induced by the augmentation $s\colon H_X \to \mathscr{O}_X$ coming from the zero map $\widetilde{T}_X \to \mathscr{O}_X$. That is, we "forget" the structure of $\mathscr{O}_X$-module to an $H_X$ module structure via $s\colon H_X \to \mathscr{O}_X$. Since the composite $\mathscr{O}_X \to H_X \xrightarrow{s} \mathscr{O}_X$ is the identity, we see that the underlying $\mathscr{O}_X$-module of $s_*\mathscr{E}$ is just $\mathscr{E}$ itself. This is the right adjoint to the forgetful functor $\mathtt{Perf}(H_X) \to \mathtt{Perf}(X)$. (Intuitively $s_*\mathscr{E}$ just endows $\mathscr{E}$ with the zero Higgs field. Geometrically, this corresponds to derived Higgs bundles supported on the zero section of the cotangent bundle.)

Similarly, this is a closed symmetric monoidal structure, meaning that we have an internal hom, which also commutes with the forgetful $s_*$. This structure now passes down to Higgs bundles.

Proposition 2.2.8. Let $X$ be a smooth rigid-analytic variety over $C$. The category of Higgs bundles admit a closed symmetric monoidal structure making the forgetful funtor and the zero inclusion

$$\pi_*\colon \mathscr{H}\mathrm{iggs}(X) \to \mathtt{Perf}(X), \quad s_*\colon \mathtt{Perf}(X) \hookrightarrow \mathscr{H}\mathrm{iggs}(X)$$

have a canonical closed symmetric monoidal functor structure.

Proof. We must show that the symmetric monoidal structure on $\mathsf{Perf}(H_X)$ coming from the Hopf algebra structure of $H_X$ preserves $\mathsf{Higgs}(X)$. But this follows from the fact that the tensor and hom commute with $\pi_*$ and the fact that $\mathsf{Perf}(X)$ is preserved under these. □

We can now give a more geometric explanation for the functors $\mathscr{A}$ and $R\Gamma_{\mathtt{Dol}}$ defined in the last section. As usual, cohomology can be understood as morphisms out of the tensor unit.

Definition 2.2.9. Let $X$ be a smooth rigid-analytic variety over $C$. If $\mathscr{E}$ is a derived Higgs bundle on $X$, we define

$$\mathscr{A}(X,\mathscr{E}) = \pi_* R\underline{\mathtt{Hom}}_{H_X}(\mathscr{O}_X,\mathscr{E}) \in \mathscr{D}(\mathscr{O}_X), \quad R\Gamma_{\mathtt{Dol}}(X,\mathscr{E}) = R\Gamma_{\mathtt{\acute{e}t}}\mathscr{A}(X,\mathscr{E}),$$

where $R\underline{\mathtt{Hom}}_{H_X}(\mathscr{E},\mathscr{V})$ denotes the internal hom computed in $\mathsf{Perf}(H_X)$.

We observe that, since $s_*$ is symmetric monoidal, the object $\mathscr{O}_X$ inherts a canonical $\mathbf{E}_\infty$-algebra structure. Hence, both functors above inhert a lax-monoidal structure.

Proposition 2.2.10. Let $E \in \mathsf{Higgs}(X)$ be a Higgs bundle on $X$. Then the two definitions of $\mathscr{A}$ and $R\Gamma_{\mathtt{Dol}}$ agree.

Proof. Since the inclusion of Higgs bundles in $\mathscr{H}\mathrm{iggs}(X)$ is symmetric monoidal, it suffices to prove the corollary. Now the proof follows as in [4, Cor. 6.22] from the Koszul resolution of the zero section $H_X \twoheadrightarrow \mathscr{O}_X$. This is a resolution

$$\mathscr{O}_X \cong \left[\widetilde{T}_X^d \otimes_{\mathscr{O}_X} H_X \to \cdots \to \widetilde{T}_X \otimes_{\mathscr{O}_X} H_X \xrightarrow{\mu} H_X\right]$$

of $\mathscr{O}_X$ by locally free $H_X$-modules. Here $\widetilde{T}_X^d = (\widetilde{\Omega}_X^d)^\vee$, and $\mu$ is the structure map of the $\mathscr{O}_X$-algebra $H_X$.

We therefore conclude that $R\underline{\mathtt{Hom}}_{H_X}(\mathscr{O}_X,E) \cong \mathscr{A}(E,\theta)$ by analysing the dual differentials. We finish the proof by taking $R\Gamma_{\mathtt{\acute{e}t}}$ on both sides. □

We note that the definition above gives us a canonical lax-monoidal structure on the functors $\mathscr{A}$ and $R\Gamma_{\mathtt{Dol}}$.

Corollary 2.2.11. The lax structure endows $\mathscr{A}(X,\mathscr{O}_X)$ with an $\mathbf{E}_\infty$-algebra struture. By the proposition above, we can write

$$\mathscr{A}(X,\mathscr{O}_X) = \bigoplus \widetilde{\Omega}^i[-i],$$

and we can recognize this to be the algebra structure coming from the wedge products

$$\wedge : \widetilde{\Omega}^i[-i] \otimes \widetilde{\Omega}^j[-j] \to \widetilde{\Omega}^{i+j}[-i-j].$$

§2.3. Unipotent Higgs bundles In this section we analyse the unipotent (and derived unipotent) Higgs bundles. We note that this definition makes use of only the geometry of our space, and it is remarkable that it will turn out to depend only on the étale homotopy type (or more precisely the étale fundamental groupoid) of our space.

We finish this section by relating unipotent Higgs bundles with modules over the $\mathbf{E}_\infty$-algebra $\mathscr{A}(X,\mathscr{O}_X)$.

Definition 2.3.1. The category of unipotent Higgs bundles, denoted $\mathsf{Higgs}(X)^{\mathrm{uni}}$, is the smallest full subcategory of $\mathsf{Higgs}(X)$ containing $\mathscr{O}_X$ and closed under extensions. The category of derived unipotent Higgs bundles $\mathscr{H}\mathrm{iggs}(X)^{\mathrm{uni}}$ is smallest stable subcategory of $\mathscr{H}\mathrm{iggs}(X)$ spanned by the unit.

A Higgs bundle in $\mathsf{Higgs}(X)^{\mathrm{uni}}$ is said to be a unipotent. That is, $\mathscr{E}$ is unipotent if there exists a filtration

$$0 \subset \mathscr{E}_1 \subset \mathscr{E}_2 \cdots \subset \mathscr{E}_r = \mathscr{E}$$

by sub-Higgs bundles whose graded pieces are isomorphic to $\mathscr{O}_X$ as Higgs bundles.

Remark 2.3.2. As the quasi-pro-étale vector bundle case, we have an inclusion

$$\mathsf{Higgs}(X)^{\mathrm{uni}} \subset \mathscr{H}\mathrm{iggs}(X)^{\mathrm{uni}}.$$

Clearly, the unit $\mathscr{O}_X$ is unipotent. Also the category of unipotent Higgs bundles is closed under extensions in $\mathsf{Higgs}(X)$, and in particular under direct sums.

Remark 2.3.3. A rank one Higgs bundle that is unipotent must be isomorphic to the trivial Higgs bundle $\mathscr{O}_X$. In this sense, unipotent Higgs bundles are orthogonal to Higgs line bundles.

However, if $X$ is a smooth affinoid then any coherent sheaf with zero Higgs field is derived unipotent. Indeed, by the descent results of [2] we can identify vector bundles and perfect complexes of $\mathscr{O}_X$-modules with vector bundles and perfect complexes of $R$-modules. Using regularity we can now find a resolution of any vector bundle by free modules, which implies the result.

Proposition 2.3.4. Let $X$ be a smooth rigid-analytic variety over $C$. Then the category of (derived) unipotent Higgs bundles inherits the closed symmetric monoidal structure from (derived) Higgs bundles.

Proof. The unit $\mathcal{O}_X$ is (derived) unipotent. Let $E$ be a unipotent Higgs bundle Consider the category of all Higgs bundles $F$ such that $E \otimes F$ is unipotent. Then $\mathcal{O}_X$ is in this category and since vector bundles are flat, and unipotent vector bundles are closed under extensions, this coincides with $\mathsf{Higgs}(X)^{\mathtt{uni}}$. The same argument works to see that the internal Hom is also preserved, since $\underline{\mathtt{Hom}}(E,\_)$ is exact when $E$ is a vector bundle.

For the derived case a similar argument shows that for any unipotent derived Higgs bundle $\mathscr{E}$ the category of derived Higgs bundles $\mathscr{F}$ such that $\mathscr{E} \otimes \mathscr{F}$ is derived unipotent is stable, and similarly for the internal Hom. □

Example 2.3.5. Let $X$ be a proper rigid analytic variety with $\mathrm{H}^1(X,C) = 0$ (for example anything simply-connected such as $\mathbf{P}^n$ or a K3 surface). Then the category of unipotent Higgs bundles is trivial in the sense that it is equivalent to the category of finite dimensional $C$ vector spaces.

Indeed, by the Hodge decomposition (Corollary 3.3.6) we have

$$\mathrm{H}^1(X,C) \cong \mathrm{H}^1(X,\mathcal{O}_X) \oplus \mathrm{H}^0(X,\widetilde{\Omega}^1)$$

so both groups on the right vanishes. The first vanishing says that all vector bundle extensions of $\mathcal{O}_X$ by $\mathcal{O}_X$ are trivial, so if $\mathscr{E}$ is unipotent then it is isomorphic to $\mathcal{O}_X^n$ as a vector bundle. The second implies that the Higgs field is zero, since there are no globally defined 1-forms.

This example shows how unipotent Higgs bundles are intimately linked with the fundamental group of the variety. This motivates a Tannakian study of the category of unipotent Higgs bundles.

Proposition 2.3.6. Let $X$ be a proper, connected adic space over $C$. The category of unipotent Higgs bundles is an abelian, rigid, tensor category. Fixing a point $x \in X(C)$, then we also have a canonical fiber functor

$$F_x \colon \mathsf{Higgs}_{\mathtt{uni}}(X) \to C\text{-}\mathsf{Vect}$$

sending $\mathscr{E}$ to $\mathscr{E}_x$, which makes it into a Tannakian category.

Properness is essential for the theorem to work. If $s \colon \mathcal{O}_X \to \mathcal{O}_X$ is any section, then the kernel in the category of unipotent bundles has to be 0 or $\mathcal{O}_X$.

Lemma 2.3.7. Let $X/C$ be a proper rigid-analytic variety[3]. The category of unipotent vector bundles, ie. the full subcategory of $\mathsf{VB}(X)$ consisting of successive extensions of $\mathcal{O}_X$, is abelian.

Proof. The proof of [20, Chp. 4, Lemma 2] applies mutatis mutandis. □

Proof (of Proposition 2.3.6). It follows straight from the lemma that the category of unipotent Higgs bundles is abelian, since the kernel and cokernel will have canonical Higgs fields. We've already seen in proposition 2.3.4 that unipotent Higgs bundles are closed under the symmetric monoidal strucure. (In particular they form a rigid tensor category.)

Finally, the fiber functor is exact and faithful. Exactness is clear since any exact sequence of vector bundles splits on a neighbourhood of $x$. Faithfulness now follows from exactness and the fact that the category of unipotent vector bundles on proper spaces is abelian. Namely if a morphism $f \colon E \to E'$ is non-zero, its kernel is now a vector bundle of rank strictly less then the rank of $E$, so $f_x$ is non-zero. □

[3] Or more generally a locally ringed topos over a field $k$ with $\mathrm{H}^0(X,\mathcal{O}_X) = k$.

Using the Tannakian reconstruction theorem we get the following definition, whose name will be justified in Corollary 3.4.5.

Definition 2.3.8 (The unipotent fundamental group). Let $X$ be a proper, connected adic space over $C$, and fix a base point $x \in X(C)$. Then the unipotent fundamental group of $X$ at $x$ is defined to be the algebraic group

$$\pi_1^{\mathtt{uni}}(X,x) = \underline{\mathrm{Aut}}^{\otimes}(F_x).$$

We have a canonical equivalence $\mathsf{Higgs}_{\mathtt{uni}}(X) \xrightarrow{\sim} \mathsf{Rep}_C(\pi_1^{\mathtt{uni}})$ and therefore this algebraic group is indeed unipotent (since the regular representation is unipotent).

We can now relate unipotent Higgs bundles and $\mathscr{A}(X,\mathscr{O}_X)$-modules.

Proposition 2.3.9. The lax-monoidal functor $\mathscr{A} : \mathscr{H}\mathrm{iggs}(X) \to \mathscr{D}(\mathscr{O}_X)$ induces a strong monoidal equivalence

$$\mathscr{H}\mathrm{iggs}(X)^{\mathtt{uni}} \xrightarrow{\sim} \mathscr{A}(X,\mathscr{O}_X)\text{-}\mathsf{Mod}^{\mathtt{uni}},$$

where the right hand side is seen as the stable $\infty$-category of modules over the $\mathbf{E}_\infty$-algebra $\mathscr{A}(X,\mathscr{O}_X)$ (Corollary 2.2.11).

The full subcategory $\mathsf{Higgs}(X)^{\mathtt{uni}} \subset \mathscr{H}\mathrm{iggs}(X)^{\mathtt{uni}}$ is identified with the smallest subcategory of the right hand side which contains the unit and is closed under fiber sequences.

Proof. This is essentially the same proof as Proposition 1.2.5. Namely, we see that $\mathscr{A}$ is symmetric monoidal by fixing an $\mathscr{E}$ and considering the subcategory of $\mathscr{V}$ such that the lax structure is an isomorphism. Then the same argument shows this category contains to be stable (since we only use that tensoring and $\mathscr{A}$ is exact). Now since every object is dualizable, fully-faithfulness of $\mathscr{A}$ reduces to the isomorphism

$$\mathrm{Hom}(\mathscr{O}_X,\mathscr{E}) \cong \mathrm{Hom}_{\mathscr{A}(X,\mathscr{O}_X)}(\mathscr{A}(X,\mathscr{O}_X),\mathscr{E}).$$

which is clear. □

Remark 2.3.10. One can show, using the same proof, that unipotent Higgs bundles correspond to $R\Gamma_{\mathtt{Dol}}(X,\mathscr{O}_X)$-modules, that is,

$$R\Gamma_{\mathtt{Dol}} : \mathscr{H}\mathrm{iggs}(X)^{\mathtt{uni}} \xrightarrow{\sim} R\Gamma_{\mathtt{Dol}}(X,\mathscr{O}_X)\text{-}\mathsf{Mod}^{\mathtt{uni}}$$

is a symmetric monoidal equivalence. In fact the same methods also show that the functor

$$R\Gamma_{\mathtt{\acute{e}t}} : \mathscr{A}(X,\mathscr{O}_X)\text{-}\mathsf{Mod}^{\mathtt{uni}} \xrightarrow{\sim} R\Gamma_{\mathtt{Dol}}(X,\mathscr{O}_X)\text{-}\mathsf{Mod}^{\mathtt{uni}}$$

is an equivalence.

§2.4. Locally unipotent Higgs bundles

Unipotent Higgs bundles do not satisfy descent for the étale topology; for example any line bundle is locally unipotent but $\mathscr{O}_X$ is the only unipotent Higgs bundle of rank 1. A more local version of this definition is the following.

Definition 2.4.1. A Higgs bundle is said to be locally unipotent if it unipotent after pullback to an étale cover of $X$. The category of locally unipotent Higgs bundles is denote by $\mathsf{Higgs}(X)^{\mathtt{l.uni}}$.

Similarly a derived Higgs bundle $\mathscr{E}$ is said to be locally derived unipotent if there exists an étale cover $f: Y \to X$ such that $\mathscr{E}_Y$ is unipotent. The full subcategory of $\mathscr{H}\mathrm{iggs}(X)$ generated by locally unipotent bundles is denoted $\mathscr{H}\mathrm{iggs}(X)^{\mathtt{l.uni}}$.

There is a fully faithful functor $\mathsf{VB}(X_{\mathrm{an}}) \hookrightarrow \mathsf{Higgs}(X)^{\mathtt{l.uni}}$ given by endowing a vector bundle with the zero field. This category is usually not abelian (for example the morphism $\mathscr{O} \to \mathscr{O}(1)$ in $\mathbf{P}^1$ has no kernel, as the forgetful morphism would preserve it). A class of examples, due to Simpson, are those Higgs bundles which are fixed under the $\mathbf{G}_m$-action $t(E,\theta) = (E, t\theta)$, that is, those bundles for which there is an isomorphism

$$f: (E,\theta) \xrightarrow{\sim} (E, t\theta)$$

for some $t \in C^\times$.

Proposition 2.4.2. Let $X$ be a smooth rigid-analytic space over $C$. The association $Y \mapsto \mathsf{Higgs}(Y)^{\mathtt{l.uni}}$ and $Y \mapsto \mathscr{H}\mathrm{iggs}(Y)^{\mathtt{l.uni}}$ becomes a stack on the small étale site $X_{\mathtt{\acute{e}t}}$ of $X$. The canonical natural transformations

$$\mathsf{Higgs}(Y)^{\mathtt{uni}} \to \mathsf{Higgs}(Y)^{\mathtt{l.uni}}, \quad \mathscr{H}\mathrm{iggs}(Y)^{\mathtt{uni}} \to \mathscr{H}\mathrm{iggs}(Y)^{\mathtt{l.uni}},$$

identify the latter as a sheafification of the former.

Proof. By descent for (derived) Higgs bundles, we can see that locally (derived) unipotent Higgs bundles form a stack. Indeed, a descent data along a cover $Y \to X$ glues uniquely to a Higgs bundle $E$ and $E_Y$ is locally unipotent hence so is $E$. Any morphism from $\mathsf{Higgs}(\_)^{\mathtt{uni}}$ into a sheaf factors locally unipotent sheaves by gluing, so this is indeed the sheafification. □

Although we will not need the characterizations below, one can relate the notion of unipotent Higgs bundles with those whose Higgs field is nilpotent.

Definition 2.4.3. Let $X$ be a smooth rigid-analytic variety and let $(E,\theta)$ be a Higgs bundle over $X$. We say that $E$ is nilpotent if $\theta$ is, in the sense that the image of

$$\widetilde{T}^1_X \to \mathrm{End}(E)$$

lies in the nilpotent endomorphisms of $E$.

Some remarks about the definition. To check this condition we can locally trivialize $\widetilde{T}^1$ which yields us $d$ operators

$$\theta_1, \dots, \theta_d : E \to E$$

on $E$. Then the definition above says that these operators are nilpotents. One may worry that this depends on trivialization, but since the $\theta_i$ commute any linear combination of the $\theta_i$ is nilpotent also. Alternatively a Higgs bundle is nilpotent if the composite

$$E \to E \otimes \Omega^1 \to \cdots \to E \otimes \widetilde{\Omega}^1 \otimes \cdots \otimes \widetilde{\Omega}^1$$

is eventually zero.

Being nilpontent is already local for the étale topology, since one can check being zero on an étale cover, and the index of nilpotency cannot surpass the rank of the bundle in question (by Nakayama's lemma and the case of fields). Geometrically these bundles are those supported on a formal completion of the zero section of the cotangent bundle.

Proposition 2.4.4. Let $X$ be a smooth rigid-analytic space and $(E,\theta)$ be a Higgs bundle. Then the following hold:

i. If $E$ is locally unipotent then it is nilpotent.

ii. If $E$ is nilpotent, then it is locally derived unipotent.

iii. If $\dim X = 1$ and $E$ is nilpontent, then it is locally unipotent.

Proof. For i), we see that the trivial Higgs bundle is nilpotent and an extension of nilpotent endomorphisms is nilpotent. For ii) we pass to an open affinoid cover trivializing the tangent bundle as above and write $\theta_i$ for the nilpotent endomorphisms of $E$. Since they commute we see that $K = \bigcap \ker_i \theta_i$ is a non-trivial sub-Higgs sheaf with zero Higgs field. Passing to the quotient $E/K$ we get another Higgs sheaf with nilpotent Higgs field, and this process must end by finite generation. By regularity we see that any Higgs sheaf with zero Higgs field is locally derived unipotent, and hence so is $E$ by induction.

For iii), let $E$ be a nilpotent Higgs bundle. Consider as above the subsheaf $K \subset E$ which, on an affinoid $\mathrm{Spa}(A, A^+)$ trivializing $T^1$ and $E$, is given by the kernel of the Higgs field. Since the ambiguity of generator does not change the kernel $K$, we see that $K$ is a well defined sub-Higgs sheaf of $E$. We claim that $K$ is furthermore a sub-bundle, meaning that $E/K$ is locally (finite) projective.

But by regularity and our dimension assumption we see that $A$ is a principal ideal domain and hence $E/K \subset E$ is free, being a submodule of a free module. Hence by induction we see that $E$ is unipotent. □

Remark 2.4.5. We suspect that point iii) above is not valid in higher dimensions, even assuming properness. The problem seem to be related to the stronger problem of finding a subbundle $E_0$ of a nilpotent bundle $E$ (ie. with $E/E_0$ also locally free) whose Higgs field is 0. One can construct examples where the kernel of the Higgs field is not a subbundle.

# 3 The correspondence

We want to relate unipotent quasi-pro-étale vector bundles and unipotent Higgs bundles. We've established in Propositions 2.3.9 and 1.2.5 that we can relate these to unipotent objects in the category of $Rv_*\widehat{\mathcal{O}}_X$-modules and $\mathscr{A}(X, \mathcal{O}_X)$-modules respectively.

Therefore we seek to construct an isomorphism

$$\mathscr{A}(X,\mathcal{O}_X) \xrightarrow{\sim} R\nu_*\widehat{\mathcal{O}}_X$$

of $\mathbf{E}_\infty$-algebras which will realize the unipotent Simpson correspondence.

The sheaf of $\mathbf{E}_\infty$-algebras $R\nu_*\widehat{\mathcal{O}}_X$ admits a filtration, the Hodge-Tate filtration, which for $X$ proper, in view of the primitive comparison theorem, induces a filtration on the étale cohomology of $X$. This filtration is analogous to the Hodge filtration in complex geometry.

The proof of the correspondence will then be given in the following steps below. We observe that we are essentially just reproving the Hodge-Tate decomposition.

- Showing that the associated graded $\mathrm{gr}_{\mathrm{HT}} R\nu_*\widehat{\mathcal{O}}_X$ is the symmetric $\mathbf{E}_\infty$-algebra on $\widetilde{\Omega}^1_X[-1]$ (Theorems 3.1.7 and 3.2.2).
- Showing that $\mathscr{A}(X,\mathcal{O}_X)$ is the free $\mathbf{E}_\infty$-algebra on $\widetilde{\Omega}^1_X$, and hence is isomorphic to the associated graded of the Hodge-Tate filtration (Corollary 3.1.3).
- Showing that the Hodge-Tate filtration splits canonically as soon as we fix a deformation $\widetilde{X}/(\mathbf{B}^+_{\mathrm{dR}}/\xi^2)$ (Corollary 3.3.2), which exists canonically for varieties defined over a finite extension of $\mathbf{Q}_p$, or non-canonically for compactifiable or affinoid $C$-varieties.

Finally we remark that we are working over an algebraically closed field $C$ just for convenience. The reader may replace $C$ by an arbitrary mixed characteristic perfectoid field containing all $p$-power roots of unity. Such assumption goes back to $p$-adic Hodge theory and is used implicitly in Theorem 3.1.7 and Theorem 3.2.2.

## §3.1. The Hodge-Tate filtration

In this subsection we defined the Hodge-Tate filtration on pro-étale cohomology. We start by showing an important computation in group cohomology that implies that both the associated graded of $R\nu_*\widehat{\mathcal{O}}_X$ and $\mathscr{A}(X,\mathcal{O}_X)$ are free $\mathbf{E}_\infty$-algebras. This will allow us to easily define maps of $\mathbf{E}_\infty$-algebras via the universal property.

In the lemma below, we will denote $\mathtt{Sym}(M[-1])$ by $\mathtt{Sym}\, M[-1]$ to clean up the notation. This is a coconnective analogue of a theorem of Illusie (which asserts the same but for $M[1]$ instead).

Lemma 3.1.1. Let $M$ be a finite locally free $\mathcal{O}_X$-module on $X_{\text{ét}}$. Then there is an equivalence of $\mathbf{E}_\infty$-algebras

$$\mathtt{Sym}\, M[-1] \cong \bigoplus_n \bigwedge^n M[-n],$$

where the right hand side is endowed with the usual wedge product.

In other words, the above implies that $\mathtt{Sym}\, M[-1]$ is formal, that is, equivalent to the direct sum of its cohomology sheaves (equivalent, it can be represented by a complex with zero differential).

Proof. This is a local statement, so we can suppose that $M$ is free, and since both sides are take direct sums to coproducts (of $\mathbf{E}_\infty$-algebras), we can also reduce to the case of rank 1. We can also do this computation at the level of pre-sheaves, since it will be clear a posteriori that the result is a sheaf.

We consider a $\mathbf{Q}$-algebra $R$, and construct an isomorphism $\mathrm{Sym} R[-1] \cong R \oplus R[-1]$. By [17, Ex. 3.1.3.14], for any ring $R$ we have $\mathrm{Sym} R[-1] \cong \bigoplus \mathrm{Sym}^n R[-1]$, with

$$\mathrm{Sym}^n R[-1] = (R^{\otimes n}[-n])_{h\Sigma_n} = (R^{\otimes n})_{h\Sigma_n}[-n].$$

We note that $R^{\otimes n} \cong R$ as an $R$-module, but $\Sigma_n$ acts via the sign action.

To finish, we note that in characteristic 0 we have that $R_{h\Sigma_n} = 0$ for the sign action on $R$. Indeed, our assumptions imply

$$\mathrm{H}_i(R_{h\Sigma_n}) = \mathrm{H}_i(\Sigma_n, R) = 0, \quad i > 0,$$

since $\Sigma_n$ is a finite group. Finally $\mathrm{H}_0(R_{h\Sigma_n}) = R_{\Sigma_n} = R/2R = 0$ since 2 is invertible in $R$. □

Remark 3.1.2. For the above result to be true, it is crucial that we are in characteristic 0. If 2 is not invertible then $R_{\Sigma_n}$ might be non-zero. Furthermore, the group homology of $\Sigma_n$ is rather non-trivial with $\mathbf{Z}$ or $\mathbf{F}_p$ coefficients!

Corollary 3.1.3. Let $X$ be a smooth, rigid-analytic variety over $C$. Then there is a canonical isomorphism of $\mathbf{E}_\infty$-algebras

$$\mathrm{Sym}\, \widetilde{\Omega}^1[-1] \cong \bigoplus_n \widetilde{\Omega}^n_X[-n] = \mathscr{A}(X, \mathscr{O}_X).$$

Proof. Lemma above and Corollary 2.2.11. □

Given a map $\widetilde{\Omega}^1[-1] \to R$ in $\mathscr{D}(X)$, with $R$ in $\mathsf{CAlg}(\mathscr{D}(X))$, one can explicitly define the homotopy class of the induced morphism $\mathrm{Sym}\, \widetilde{\Omega}^1[-1] \to R$ (See for example the proof of [13, Prop. 7.2.5]). Since we're in characteristic 0, the canonical map $(\widetilde{\Omega}^1)^{\otimes n} \to \widetilde{\Omega}^n$ admits a section

$$\omega_1 \wedge \cdots \wedge \omega_n \mapsto \frac{1}{n!} \sum_{\sigma \in \Sigma_n} \mathrm{sgn}(\sigma)\, \omega_{\sigma(1)} \otimes \cdots \otimes \omega_{\sigma(n)};$$

this allows us to define the induced map as the direct sum of the maps

$$\widetilde{\Omega}^n[-n] \to (\widetilde{\Omega}^1[-1])^{\otimes n} \to R^{\otimes n} \to R.$$

In other words, we must now show that $R^1\nu_*\widehat{\mathscr{O}}_X$ generates $\mathrm{gr}_{\mathrm{HT}} R\nu_*\widehat{\mathscr{O}}_X$ as a graded ring via the cup product. (This also works for any $M$ in the lemma above).

Definition 3.1.4 (The Hodge-Tate filtration). Let $X$ be a smooth rigid-analytic variety over $C$ and let $\nu\colon X_{\mathrm{qpro\acute{e}t}} \to X_{\mathrm{\acute{e}t}}$ be the canonical map of sites. Then the Hodge-Tate filtration on $R\nu_*\widehat{\mathscr{O}}_X$ is defined to be the canonical filtration on it, that is, the filtration induced by $\tau^{\leq i} R\nu_*\widehat{\mathscr{O}}_X$. The associated graded of $R\nu_*\widehat{\mathscr{O}}_X$ is therefore

$$\mathrm{gr}_{\mathrm{HT}} R\nu_*\widehat{\mathscr{O}}_X = \bigoplus_i R^i\nu_*\widehat{\mathscr{O}}_X[-i].$$

Remark 3.1.5. The above definition still makes sense for all proper rigid analytic varieties $X$ over $C$, however, it is only well behaved for $X$ smooth. In general, one can use the tools of Kan extensions to reduce to the smooth case. This is made precise using the Éh topology in [13].

Remark 3.1.6. The above induces a filtration, also called the Hodge-Tate filtration, on the pro-étale cohomology $R\Gamma_{\text{ét}}(X, R\nu_*\widehat{\mathscr{O}}_X) = R\Gamma_{\text{qproét}}(X, \widehat{\mathscr{O}}_X)$ of $X$. This descends to a filtration on cohomology groups $\mathrm{H}^k_{\text{qproét}}(X, \widehat{\mathscr{O}}_X)$ via the image of $\mathrm{H}^k_{\text{ét}}(X, \tau^{\leq i} R\nu_*\widehat{\mathscr{O}}_X)$. If the filtration splits, then in fact we have injections

$$\dots \mathrm{H}^k_{\text{ét}}(X, \tau^{\leq i} R\nu_*\widehat{\mathscr{O}}_X) \subset \mathrm{H}^k_{\text{ét}}(X, \tau^{\leq i+1} R\nu_*\widehat{\mathscr{O}}_X) \subset \cdots \subset \mathrm{H}^k_{\text{qproét}}(X, \widehat{\mathscr{O}}_X)$$

and the associated graded will be $\mathrm{H}^k_{\text{ét}}(X, R^i\nu_*\widehat{\mathscr{O}}_X[-i]) = \mathrm{H}^{k-i}_{\text{ét}}(X, R^i\nu_*\widehat{\mathscr{O}}_X)$.

We can now compute the associated graded of the Hodge-Tate filtration. All the essential ideas of the proof are not new, and we follow closely the ideas of [6], [7] and [13].

Theorem 3.1.7. Let $X$ be a smooth rigid-analytic variety over $C$. Then $R^0\nu_*\widehat{\mathscr{O}}_X \cong \mathscr{O}_X$, $R^1\nu_*\widehat{\mathscr{O}}_X$ is a vector bundle of rank equal to the dimension of $X$ and the induced map

$$\mathtt{Sym}(R^1\nu_*\widehat{\mathscr{O}}_X[-1]) \xrightarrow{\sim} \mathrm{gr}_{\mathrm{HT}} R\nu_*\widehat{\mathscr{O}}_X = \bigoplus_n R^n\nu_*\widehat{\mathscr{O}}_X[-n]$$

is an equivalence of $\mathbf{E}_\infty$-algebras. Here $\mathtt{Sym}\, M$ denotes the free $\mathbf{E}_\infty$-algebra on $M \in \mathscr{D}(X)$.

Proof. We have a map $R^1\nu_*\widehat{\mathscr{O}}_X[-1] \to \mathrm{gr}_{\mathrm{HT}} R\nu_*\widehat{\mathscr{O}}_X$ given by the inclusion of the degree 1 part. By the lemma and the above computation, we need to show that $R^i\nu_*\widehat{\mathscr{O}}_X = \bigwedge^i R^1\nu_*\widehat{\mathscr{O}}_X$ and that $R^1\nu_*\widehat{\mathscr{O}}_X$ generates the whole cohomology ring via cup product.

The statement is étale local so we can suppose that $X$ is an affinoid admitting an étale map to a rigid-analytic torus

$$X \to \mathbf{T}^n = \mathrm{Spa}\, C\langle T_1^{\pm 1}, \dots, T_d^{\pm 1}\rangle.$$

Now, the torus admits a quasi-pro-étale cover by an affinoid perfectoid

$$\widetilde{\mathbf{T}}^n = \mathrm{Spa}\, C\left\langle T_1^{\pm 1/p^\infty}, \dots, T_d^{\pm 1/p^\infty}\right\rangle \to \mathbf{T}^n$$

and therefore so does $X$ by base change. That is, we have a perfectoid affinoid cover $\widetilde{X} = \widetilde{\mathbf{T}}^n \times_{\mathbf{T}^n} X \to X$ of $X$ which, by the acyclicity of $\widehat{\mathscr{O}}_X$ (Theorem B.0.12), can be used to compute the cohomology of $\widehat{\mathscr{O}}_X$ as Čech cohomology on global sections. Now $\widetilde{\mathbf{T}}^n \to \mathbf{T}^n$ (and hence $\widetilde{X} \to X$) is a $\mathbf{Z}_p(1)$ torsor, since it is a limit of the $\mathrm{Spa}\, C\langle T^{1/p^n}\rangle$ which are $\mathbf{Z}/p^n(1)$ torsors [4].

Writing $X = \mathrm{Spa}(R, R^+)$ and $\widetilde{X} = \mathrm{Spa}(R_\infty, R_\infty^+)$, We conclude that this Čech cohomology complex is just the continuous cohomology

$$R\Gamma_{\text{qproét}}(X, \widehat{\mathscr{O}}_X) \cong R\Gamma_{\mathtt{cont}}(\mathbf{Z}_p(1), R_\infty)$$

for this action of $\mathbf{Z}_p(1)$ on $R_\infty$. This computation is carried out on an integral level in [22, Lemmas 4.5, 5.5], which implies $R^1\nu_*\widehat{\mathscr{O}}_X$ is free and generates the higher pushfowards via cup products. □

[4] A word of warning on a potentially confusing abuse of notation. We have introduced a pro-finite-étale universal cover $\widetilde{X}$ of $X$, but this is not the same as the $\widetilde{X}$ defined above. Instead, the construction above is much simpler as it only captures the pro-$p$ part of the fundamental group, which is enough for the computation.

Corollary 3.1.8. The Hodge-Tate filtration on $R\nu_*\widehat{\mathscr{O}}_X$ splits if and only if the induced filtration on $\tau^{\leqq 1}R\nu_*\widehat{\mathscr{O}}_X$ splits.

Proof. Indeed, if we have a splitting $R^1\nu_*\widehat{\mathscr{O}}_X[-1] \to \tau^{\leqq 1}\widehat{\mathscr{O}}_X$ we can compose with the filtration map to obtain a map

$$\mathtt{Sym} R^1\nu_*\widehat{\mathscr{O}}_X[-1] \xrightarrow{\sim} R\nu_*\widehat{\mathscr{O}}_X$$

which is an equivalence, because it becomes as equivalence on associated graded. By the theorem above, the Hodge-Tate filtration splits. The converse is immediate. □

## §3.2. The lift to $\mathbf{B}_{\mathrm{dR}}^+/\xi^2$

The non-abelian Hodge correspondence depends on a flat lift $\widetilde{X}$ of $X$ to the ring of periods $\mathbf{B}_{\mathrm{dR}}^+/\xi^2$, that is, to $\mathrm{Spa}(\mathbf{B}_{\mathrm{dR}}^+/\xi^2, \mathbf{A}_{\mathtt{inf}}/\xi^2)$. In this subsection we explore a link between the pro-étale cohomology of $\widehat{\mathscr{O}}_X$ and the cotangent complex, which finishes our computation of the associated graded $\mathrm{gr}_{\mathtt{HT}} R\nu_*\widehat{\mathscr{O}}_X$ and allows us to split this filtration when the aforementioned lift exists.

For ease of notation, we will also denote $\mathbf{B}_{\mathrm{dR}}^+/\xi^2$ by $B_2$ in the computations.

Definition 3.2.1. Let $X$ be an adic space over $C$. A (flat) lift to $\mathbf{B}_{\mathrm{dR}}^+/\xi^2 = B_2$ (or a deformation to $\mathbf{B}_{\mathrm{dR}}^+/\xi^2$) is a cartesian square

$$\begin{array}{ccc} X & \longrightarrow & \widetilde{X} \\ \downarrow & & \downarrow \\ \mathrm{Spa}\, C & \longrightarrow & \mathrm{Spa}\, B_2 \end{array}$$

with $\widetilde{X} \to B_2$ flat. Here, the positive de Rham ring is given its canonical topology coming from the $p$-adic topology of $\mathbf{A}_{\mathtt{inf}}/\xi^2$. In other words, a lift to $\mathbf{B}_{\mathrm{dR}}^+/\xi^2$ is an adic space $\widetilde{X}$ flat over $\mathbf{B}_{\mathrm{dR}}^+/\xi^2$ together with an identification of $X$ with the zero locus of $\xi$.

The deformation theory of $X$ is controlled by the cotangent complex $\mathrm{L}_{X/C}^{\mathrm{an}}$, which can be thought as a "complete" version of the usual (topos-theoretic) cotangent complex (see definition A.0.7).

To understand what is the role of the lift in splitting the Hodge-Tate filtration, we first remind the reader that if $X \hookrightarrow Y$ is a closed immersion given by a coherent ideal sheaf $\mathscr{I}$, then $\tau_{\leqq 1}\mathrm{L}_{Y/X}^{\mathrm{an}} \cong \mathscr{I}/\mathscr{I}^2[1]$ (see proposition A.0.11 on the appendix). This implies that

$$\tau_{\leqq 1}\mathrm{L}_{C/B_2}^{\mathrm{an}} = C(1)[1], \quad \tau_{\leqq 1}\mathrm{L}_{\widetilde{X}/X}^{\mathrm{an}} = \mathscr{O}_X(1)[1],$$

where the second isomorphism follows from flatness: indeed if $X/\mathrm{Spa}B_2$ is flat, we get an isomorphism $f^*C(1) \cong \mathscr{O}_X(1) \xrightarrow{\sim} \mathscr{I}$, where $\mathscr{I}$ is the ideal defining $X \hookrightarrow \widetilde{X}$, by applying the exact functor $f^*$ to the exact sequence $0 \to C(1) \to \mathbf{B}_{\mathrm{dR}}^+/\xi^2 \to C \to 0$ seen as sheaves on the topological space $|\mathrm{Spa}\, C| = |\mathrm{Spa} B_2|$.

Now assume that $X$ is smooth over $C$, so in particular $\mathrm{L}_{X/C}^{\mathrm{an}} = \Omega_{X/C}^1$. The transitivity fiber sequence for $X \to \mathrm{Spa}\, C \to \mathrm{Spa}\,\mathbf{B}_{\mathrm{dR}}^+/\xi^2$ now gives us a fiber sequence

$$\mathscr{O}_X(1)[1] \to \tau_{\leq 1}\mathrm{L}_{X/B_2}^{\mathrm{an}} \to \Omega_{X/C}^1 \to \mathscr{O}(1)[2]$$

(observe how the smoothness assumption allows us to truncate). This means, in particular, that there is an obstruction class $o \in \mathrm{Ext}^2(\Omega^1_{X/C}, \mathscr{O}_X(1)) = \mathrm{H}^2(X, \widetilde{T}_X)$ which vanishes precisely when this fiber sequence splits. This is equivalent to constructing an isomorphism

$$\tau_{\leqq 1}\mathrm{L}^{\mathrm{an}}_{X/B_2}(-1)[-1] = \mathscr{O}_X \oplus \widetilde{\Omega}^1_{X/C}[-1]$$

identifying the map $\mathscr{O}_X \to \tau_{\leqq 1}\mathrm{L}^{\mathrm{an}}_{X/B_2}(-1)[-1]$ with the map from the fiber sequence. The following theorem now relates the vanishing of this class and splitting the Hodge-Tate filtration. This is a convenient phrasing of a know result which we include the proof for convenience.

Theorem 3.2.2 ([7, Prop. 8.15] [13, Thm. 7.2.3]). *Let $X$ be a smooth rigid-analytic space over $C$. Then there is a functorial equivalence*

$$\tau_{\leqq 1}\mathrm{L}^{\mathrm{an}}_{X/B_2}(-1)[-1] \xrightarrow{\sim} \tau^{\leqq 1}R\nu_*\widehat{\mathscr{O}}_X$$

*in the derived category of $\mathscr{O}_X$-modules. Furthermore, the filtration induced by the fiber sequence above agrees with the Hodge-Tate filtration on the right hand side. In particular, $R^1\nu_*\widehat{\mathscr{O}}_X \cong \widetilde{\Omega}^1_X$.*

*Proof.* First, we see that $\tau_{\leqq 1}\mathrm{L}^{\mathrm{an}}_{X/B_2}$ is isomorphic to $\mathrm{L}^{\mathrm{an}}_{X/\mathbf{B}_{\mathtt{inf}}}$. This is due to the fact that $\xi$ is a non-zero divisor on $\mathbf{B}_{\mathtt{inf}}$, and hence $\mathrm{L}^{\mathrm{an}}_{C/\mathbf{B}_{\mathtt{inf}}} = \xi/\xi^2[1]$. The transitivity sequence for $X \to \mathrm{Spa}\, C \to \mathrm{Spa}\, \mathbf{B}_{\mathtt{inf}}$ yields

$$\mathrm{L}^{\mathrm{an}}_{X/\mathbf{B}_{\mathtt{inf}}} = \mathrm{Cof}(\Omega^1_{X/C}[-1] \to \mathscr{O}_X(1)[1]),$$

the cofiber of the same morphism coming from the fiber sequence above.

The proof of this theorem now relies on generalizing the cotangent compex to the quasi-pro-étale site of $X$, computing it there, and comparing to $\mathrm{L}^{\mathrm{an}}_{X/\mathbf{B}_{\mathtt{inf}}}$. We recall that $X_{\mathtt{qpro\acute{e}t}}$ has a basis of affinoid perfectoids; we can therefore define

$$\mathrm{L}^{\mathrm{an}}_{\widehat{\mathscr{O}}_X/\mathbf{B}_{\mathtt{inf}}} = \mathrm{L}^{\mathrm{an}}_{\widehat{\mathscr{O}}^+_X/\mathbf{A}_{\mathtt{inf}}}\left[\frac{1}{p}\right],$$

and $\mathrm{L}^{\mathrm{an}}_{\widehat{\mathscr{O}}^+_X/\mathbf{A}_{\mathtt{inf}}}$ to be the sheafification of the presheaf sending an affinoid perfectoid $\mathrm{Spa}(A, A^+)$ to $\mathrm{L}^{\mathrm{an}}_{A^+/\mathbf{A}_{\mathtt{inf}}}$. We remark that this cotangent complex is just the $p$-completed algebraic cotangent complex $\mathrm{L}_{A^+/\mathbf{A}_{\mathtt{inf}}}$ since perfectoid algebras are uniform.

The proof now relies on the existence of a natural comparison morphism

$$\mathrm{L}^{\mathrm{an}}_{X/\mathbf{B}_{\mathtt{inf}}} \to R\nu_*\mathrm{L}^{\mathrm{an}}_{\widehat{\mathscr{O}}_X/\mathbf{B}_{\mathtt{inf}}}$$

on the étale site $X_{\mathtt{\acute{e}t}}$ which is furthermore natural in $X$. Indeed, the data of such morphism corresponds to maps $\mathrm{L}^{\mathrm{an}}_{X/\mathbf{B}_{\mathtt{inf}}}(U) \to \mathrm{L}^{\mathrm{an}}_{\widehat{\mathscr{O}}_X/\mathbf{B}_{\mathtt{inf}}}(V)$ for $V \to U$ quasi-pro-étale morphism with $V$ an affinoid perfectoid and $U \to X$ étale; these come from the functoriality of the analytic cotangent complex discussed in the appendix.

We now proceed to compute the pro-étale cotangent complex. Fix an affinoid perfectoid $\mathrm{Spa}(A, A^+)$ in $X_{\mathtt{qpro\acute{e}t}}$. Now $A^+$ is a relatively perfect $\mathscr{O}_C$-algebra, which implies that $\mathrm{L}^{\mathrm{an}}_{A^+/\mathbf{A}_{\mathtt{inf}}} = 0$. Therefore the transitivity fiber sequence of $\mathbf{A}_{\mathtt{inf}} \to \mathscr{O}_C \to A^+$ gives us

$$\mathrm{L}^{\mathrm{an}}_{\mathscr{O}_C/\mathbf{A}_{\mathtt{inf}}} \otimes_{\mathscr{O}_C} A^+ \xrightarrow{\sim} \mathrm{L}^{\mathrm{an}}_{A^+/\mathbf{A}_{\mathtt{inf}}}$$

This left hand side is well understood; since $\mathbf{A}_{\mathtt{inf}} \twoheadrightarrow \mathscr{O}_C$ is a closed immersion with kernel generated by the regular element $\xi$ then $\mathrm{L}^{\mathrm{an}}_{C/\mathbf{B}_{\mathtt{inf}}}$ is just $(\xi)/(\xi^2)[1]$, so this tensor product is a shift of the Breuil-Kisin twist $A^+\{1\}$. By varying $A$ and sheafifying we obtain a equivalences[5]

$$\widehat{\mathscr{O}}^+_X\{1\}[1] \cong \mathrm{L}^{\mathrm{an}}_{\mathscr{O}_C/\mathbf{A}_{\mathtt{inf}}} \otimes_{\mathscr{O}_C} \widehat{\mathscr{O}}_X \xrightarrow{\sim} \mathrm{L}^{\mathrm{an}}_{\widehat{\mathscr{O}}^+_X/\mathbf{A}_{\mathtt{inf}}}; \quad \mathscr{O}_X(1)[1] \xrightarrow{\sim} \mathrm{L}^{\mathrm{an}}_{\widehat{\mathscr{O}}_X/\mathbf{B}_{\mathtt{inf}}}.$$

We have now produced a natural morphism $\mathrm{L}^{\mathrm{an}}_{X/\mathbf{B}_{\mathtt{inf}}} \to R\nu_*\widehat{\mathscr{O}}_X(1)[1]$ which we must show that identifies the left-hand side with a truncation of the right-hand side. That is, we must show that

$$\mathrm{L}^{\mathrm{an}}_{X/\mathbf{B}_{\mathtt{inf}}}(-1)[-1] \to \tau^{\leq 1} R\nu_*\widehat{\mathscr{O}}_X$$

is an equivalence. Now, this result is again local so we can again suppose that $X$ is an affinoid admitting an étale map to a torus

$$f : X \to \mathbf{T}^n$$

and therefore reduce to the torus itself: we already know that the right hand side has coherent cohomology, and in the left hand side we have $\mathrm{L}^{\mathrm{an}}_{X/\mathbf{T}^n} = 0$ since $X/\mathbf{T}^n$ is étale, which implies that the canonical map

$$f^*\mathrm{L}^{\mathrm{an}}_{\mathbf{T}^n/\mathbf{B}_{\mathtt{inf}}} \xrightarrow{\sim} \mathrm{L}^{\mathrm{an}}_{X/\mathbf{B}_{\mathtt{inf}}}$$

is an equivalence.

Since $\Omega^1_{\mathbf{T}^n}$ is also free and of the same rank, it follows from Theorem 3.1.7 that both sides are isomorphic, but we still need to check that the induced map is an isomorphism. It is clear by definition that on degree zero this is an isomorphism. For the result on $\Omega^1_X$, that is on degree one, we need to be a bit more careful, and the computation was carried integrally in [7, Sec. 8.3]. □

In particular, this theorem implies links the obstruction class with $p$-adic Hodge theory.

Corollary 3.2.3. Let $X$ be a smooth rigid-analytic variety over $C$. The obstruction class $o \in \mathrm{Ext}^2(L^{\mathrm{an}}_{X/C}, \mathscr{O}_X(1)) = \mathrm{H}^2(X, \widetilde{T}_X)$ defined above detects precisely when the Hodge-Tate filtration splits.

Corollary 3.2.4. Let $X/C$ be a smooth affinoid rigid-analytic variety, or a smooth rigid-analytic curve. Then the Hodge-Tate filtration always splits.

Proof. In both cases $\mathrm{H}^2(X, \widetilde{T}_X) = 0$. □

[5] The Breuil-Kisin twist $A^+\{1\}$ is almost isomorphic to $A(1)$, so they agree after inverting $p$.

§3.3. Splitting the Hodge-Tate filtration

Finally, we can compare the obstruction class coming from Theorem 3.2.2 to split the Hodge-Tate filtration outside of the affinoid setting. The results in the last section allow us to relate the problem of finding the spitting to the geometric problem of finding a flat deformation to $\mathbf{B}_{\mathrm{dR}}^+/\xi^2$.

We are also able to be more precise here. The vanishing of the class $o$ determines a splitting but it is not canonical. However, remembering the lift makes it so. This result is in [13, Prop. 7.1.4], but we given a different, slightly more direct proof.

Theorem 3.3.1. Let $X$ be a smooth rigid-analytic variety over $C$. Then each flat deformation of $X$ to $\mathbf{B}_{\mathrm{dR}}^+/\xi^2$ determines a canonical splitting of the Hodge-Tate filtration on $R\nu_*\widehat{\mathcal{O}}_X$.

Proof. We now need the other fiber sequence associated to the flat lift $\iota\colon X \hookrightarrow \widetilde{X}$. Namely, we consider the composition $X \hookrightarrow \widetilde{X} \to \mathrm{Spa} B_2$ which yields a fiber sequence

$$L\iota^* \mathrm{L}^{\mathrm{an}}_{\widetilde{X}/B_2} \to \mathrm{L}^{\mathrm{an}}_{X/B_2} \to \mathrm{L}^{\mathrm{an}}_{\widetilde{X}/X}$$

and since $\iota$ is a closed immersion with ideal $\mathcal{O}_X(1)$ we have $\tau_{\leq 1}\mathrm{L}^{\mathrm{an}}_{\widetilde{X}/X} = \mathcal{O}_X(1)[1]$. We claim that the induced map $\tau_{\leq 1}\mathrm{L}^{\mathrm{an}}_{X/B_2}(-1)[-1] \to \tau_{\leq 1}\mathrm{L}^{\mathrm{an}}_{\widetilde{X}/X}(-1)[-1] \to \mathcal{O}_X$ splits the inclusion of $\mathcal{O}_X \to \tau_{\leq 1}\mathrm{L}^{\mathrm{an}}_{X/B_2}(-1)[-1]$. That is, we must show that the composite (beware, this is no triangle)

$$f^*\mathrm{L}^{\mathrm{an}}_{B_2/C} \xrightarrow{\delta} \mathrm{L}^{\mathrm{an}}_{X/B_2} \xrightarrow{\alpha} \mathrm{L}^{\mathrm{an}}_{\widetilde{X}/X}$$

is an equivalence ($f$ being the structure morphism $f : X \to \mathrm{Spa}\, C$) induces an isomorphism on degree 1.

Let $\mathcal{H}$ denote the cohomology of complexes (as opposed to hypercohomology). The $\mathcal{H}_1$ are isomorphic to $\mathcal{O}_X(1)$, a line bundle, and therefore it is enough to show that the induced morphism is surjective. We see that $\delta$ is an isomorphism from its defining fiber sequence. For $\alpha$ we consider the exact sequence

$$\mathcal{H}_1(\mathrm{L}^{\mathrm{an}}_{X/B_2}) \xrightarrow{\alpha} \mathcal{H}_1(\mathrm{L}^{\mathrm{an}}_{\widetilde{X}/X}) \to \mathcal{H}_0(L\iota^*\mathrm{L}^{\mathrm{an}}_{\widetilde{X}/B_2}),$$

and it is enough to show that the last arrow are zero to see surjectivity. This is given by the differential

$$\mathcal{H}_1(\mathrm{L}^{\mathrm{an}}_{\widetilde{X}/X}) \cong \xi\mathcal{O}_{\widetilde{X}} \xrightarrow{\mathrm{d}\otimes 1} \Omega^1_{\widetilde{X}/B_2} \otimes_{\mathcal{O}_{\widetilde{X}}} \mathcal{O}_X \cong \mathcal{H}_0(L\iota^*\mathrm{L}^{\mathrm{an}}_{\widetilde{X}/B_2})$$

so it suffices to note that $\mathrm{d}\xi = 0$ since the differentials are $B_2$-linear. □

Corollary 3.3.2. Let $X$ be a smooth rigid-analytic variety over $C$. Then any flat lift $X \hookrightarrow \widetilde{X}$ to $\mathbf{B}_{\mathrm{dR}}^+/\xi^2$ induces an equivalence of $\mathbf{E}_\infty$-algebras

$$\mathrm{Sym}\,\widetilde{\Omega}^1_X[-1] \xrightarrow{\sim} R\nu_*\widehat{\mathcal{O}}_X.$$

We now tackle the problem of actually producing lifts to $\mathbf{B}_{\mathrm{dR}}^+/\xi^2$, and therefore splittings of the Hodge-Tate filtration. This following criterion follows from the description of the Galois-invariants of $\mathbf{B}_{\mathrm{dR}}^+/\xi^2$.

Lemma 3.3.3. Suppose that $X$ is defined over $K$, a finite extension of $\mathbf{Q}_p$. Then there is a canonical lift $X \hookrightarrow \widetilde{X}$ to $\mathbf{B}_{\mathrm{dR}}^+$, and hence to $\mathbf{B}_{\mathrm{dR}}^+/\xi^2$.

Proof. It suffices to find a continuous map making the diagram

$$\begin{array}{ccc} \mathbf{B}_{\mathrm{dR}}^+/\xi^2 & \longrightarrow & C \\ \uparrow & \nearrow & \\ K & & \end{array}$$

commute, since the lift will be given by the base-change from $K$ to $\mathbf{B}_{\mathrm{dR}}^+/\xi^2$, which is automatically flat.

Now this follows from the study of $\mathbf{B}_{\mathrm{dR}}^+$ and the identification of $K$ with the $G_K$-invariants of $\mathbf{B}_{\mathrm{dR}}^+$. We note that this crucially fails for $C$, as there is no continuous ring homomorphism $C \to \mathbf{B}_{\mathrm{dR}}^+$. □

Lemma 3.3.4. Let $X, Y$ be rigid spaces over $C$ and suppose there exists an étale map

$$f : Y \to X$$

and that $X$ admits a lift $X \hookrightarrow \widetilde{X}$ to $\mathbf{B}_{\mathrm{dR}}^+/\xi^2$. Then $Y$ admits a lift $Y \hookrightarrow \widetilde{Y}$ to $\mathbf{B}_{\mathrm{dR}}^+/\xi^2$ which is even étale over $\widetilde{X}$. This lift is functorial on $X_{\text{ét}}$.

Proof. This follows by the topological invariance of the étale site, since the extension $X \hookrightarrow \widetilde{X}$ is square-zero. That is, given an étale morphism $Y \to X$ there is a unique extension $\widetilde{Y} \to \widetilde{X}$ which is étale, and hence flat over $\mathbf{B}_{\mathrm{dR}}^+/\xi^2$. Functoriality follows immediately from the uniqueness of the étale lift. □

Remark 3.3.5. Using more advanced techniques one can show that, in fact, all proper (or more generally compactifiable) rigid spaces $X/C$ admit a flat deformation to $\mathbf{B}_{\mathrm{dR}}^+/\xi^2$. This is proven in [13, Thm. 7.4.4].

Corollary 3.3.6 (The Hodge-Tate decomposition). Let $X$ be a proper, smooth rigid-analytic variety over $C$. Then any lift $X \hookrightarrow \widetilde{X}$ to $\mathbf{B}_{\mathrm{dR}}^+/\xi^2$ induces a Hodge-Tate decomposition

$$\mathrm{H}^n(X, C) \cong \bigoplus_{i+j=n} \mathrm{H}^i(X, \widetilde{\Omega}_X^j). \qquad \text{(hodge)}$$

If $X$ is defined over $\mathrm{Spa}K$ for $K$ a finite extension of $\mathbf{Q}_p$ and $C = \mathbf{C}_p$, this is also equivariant for the Galois action on both sides.

Proof. Apply $R\Gamma_{\text{ét}}$ to Corollary 3.3.2 and use the primitive comparison theorem (Theorem B.0.13). The Galois equivariance follows from the canonicity of the lift for varieties defined over $K$. □

§3.4. Finishing the proof

Now we have all the ingredients for proving the unipotent Simpson correspondence. We establish in many different forms for ease of use, and draw some basic consequences of it.

Theorem 3.4.1 (The unipotent correspondence). Let $X$ be a smooth rigid-analytic space over $C$, endowed with a lift $\widetilde{X}$ to $\mathbf{B}^+_{\mathrm{dR}}/\xi^2$. There is canonical exact equivalence of symmetric monoidal abelian categories

$$\mathsf{Higgs}(X)^{\mathtt{uni}} \xrightarrow{\sim} \mathsf{VB}(X_{\mathtt{qpro\acute{e}t}})^{\mathtt{uni}};.$$

between unipotent Higgs bundles and unipotent quasi-pro-étale vector bundles. This equivalence is natural for morphisms which can be lifted to the deformations.

Similarly, under the same assumptions, there is a canonical equivalence of symmetric monoidal stable infinity categories

$$\mathscr{H}\mathrm{iggs}(X)^{\mathtt{uni}} \xrightarrow{\sim} \mathtt{Perf}(X_{\mathtt{qpro\acute{e}t}})^{\mathtt{uni}};$$

between derived unipotent Higgs bundles and derived unipotent quasi-pro-étale vector bundles. This equivalence is natural for morphisms which can be lifted to the deformations.

Proof. Follows from corollary 3.3.2 together with the results above, by noting that both the 1-categorical and derived unipotent objects are preserved under monoidal categorical equivalences, and that

$$\mathtt{Sym}\,\widetilde{\Omega}^1_X[-1] \xrightarrow{\sim} \mathscr{A}(X,\mathscr{O}_X)$$

as $\mathbf{E}_\infty$-algebras (Corollary 2.2.11 and Lemma 3.1.1). □

Corollary 3.4.2. Let $X$ be as above. There are symmetric monoidal equivalences

$$\mathsf{Higgs}(X)^{\mathtt{l.uni}} \xrightarrow{\sim} \mathsf{VB}(X_{\mathtt{qpro\acute{e}t}})^{\mathtt{l.uni}}; \quad \mathscr{H}\mathrm{iggs}(X)^{\mathtt{l.uni}} \xrightarrow{\sim} \mathtt{Perf}(X_{\mathtt{qpro\acute{e}t}})^{\mathtt{l.uni}};$$

between locally unipotent objects in each category.

Proof. If we fix a lift of $X$ to $\mathbf{B}^+_{\mathrm{dR}}/\xi^2$, then we also get a lift of each $Y \in X_{\mathtt{\acute{e}t}}$ in a compatible way by Lemma 3.3.4. It follows that the correspondence above can be improved to an equivalence of pre-stacks on $X_{\mathtt{\acute{e}t}}$, and hence an equivalence on their sheafifications. □

Corollary 3.4.3. Let $X$ be a rigid-analytic $C$-variety which admits a deformation to $\mathbf{B}^+_{\mathrm{dR}}/\xi^2$. If $(E,\theta)$ is a locally unipotent Higgs bundle on $X$, and $\widetilde{E}$ is the correponding quasi-pro-étale vector bundle, then

$$R\Gamma_{\mathtt{qpro\acute{e}t}}(X,\widetilde{E}) \cong R\Gamma_{\mathtt{Dol}}(X,E).$$

Corollary 3.4.4. Let $f : X \to Y$ be a morphism of smooth, proper, connected rigid-analytic varieties. Let also $\bar{x}$ be a geometric point of $X$ and $\bar{y} = f(\bar{x})$ and suppose that $f_* : \pi_1(X,\bar{x}) \xrightarrow{\sim} \pi_1(Y,\bar{y})$ is an equivalence. Then we have a symmetric monoidal equivalence of categories

$$\mathsf{Higgs}(Y)^{\mathtt{uni}} \xrightarrow{\sim} \mathsf{Higgs}(X)^{\mathtt{uni}}, \quad \mathscr{H}\mathrm{iggs}(Y)^{\mathtt{uni}} \xrightarrow{\sim} \mathscr{H}\mathrm{iggs}(X)^{\mathtt{uni}}.$$

That is, unipotent Higgs bundles are invariant under $\pi_1$-equivalences.

Proof. Follows by the equivalence above and 1.2.4. □

On the same vein, we can now recognise the Tannakian group associated to $\mathsf{Higgs}(X)^{\mathrm{uni}}$ as le $\pi_1$ rendu nilpotent [9]. (see also [9, §10.25] for the trancendental version of this corollary.)

Corollary 3.4.5. Let $X$ be proper, smooth connected and pointed over $C$. The unipotent fundamental group 2.3.8 is the unipotent hull of $\pi_1(X,\bar{x})$.

Proof. This is essentially by definition, see [9, Eq. 10.24.2]. □

# A The cotangent complex and deformation theory

Adic spaces admit cotangent sheaves which are defined analogously to schemes, but taking into consideration the topology of our modules (we want differentials to be continuous to talk about derivatives of analytic functions).

For simplicity we omit the superscript and write $\Omega$ intead of $\Omega^1$ in the discussion below. Also, the analytic cotangent sheaf will carry an ornament $\Omega^{\mathrm{an}}$ in this appendix for clearness, but we will drop this in the main text, since we are only dealing with adic spaces therein. The following definition is due to Huber, and it was used to give the first definition of smooth morphisms of adic spaces. We refer the reader to [16, Sec. 1.6] for proofs and further discussions of the topic.

Definition A.0.1 (Huber [16, Def.1.6.1]). Given Huber rings $A,B$ and $A \to B$ a morphism topologically of finite type, a universal derivation is a continuous map

$$\mathrm{d} : B \to \Omega^{\mathrm{an}}_{B/A}$$

into a complete topological $B$-module $\Omega_{B/A}$ which satisfies the Lebniz rule, and is universal one such, ie, any other continuous derivation onto a complete module $M$ factors uniquely as

$$\begin{array}{ccc} A & \xrightarrow{\ \mathrm{d}\ } & \Omega^{\mathrm{an}}_{B/A} \\ & \searrow \quad \swarrow & \\ & M & \end{array}$$

Universal derivations exist and commute with base change and localization as expected. In particular, we can define a cotangent sheaf $\Omega^{\mathrm{an}}_{X/Y}$ for adic spaces for tft morphisms $f : X \to Y$, and its a coherent $\mathscr{O}_X$-module. From now on we assume $X,Y$ to be rigid-analytic varieties over $K$ for simplicity.

Remark A.0.2. Of course, $\Omega^{\mathrm{an}}_{B/A}$ is to be though as a complete version of the usual cotangent sheaf. The universal property yields a natural map $\Omega_{B/A} \to \Omega^{\mathrm{an}}_{B/A}$ which is not an isomorphism in general. One shows that $\Omega^{\mathrm{an}}_{B/A}$ is the largest finitely generated $B$-module quotient of $\Omega_{B/A}$ [21, Lemma 7.2.37]. It is therefore an isomorphism whenever $B$ is a finite $A$-algebra.

Concretely, the analytic cotangent sheaf can be computed via an analogous procedure to the case of algebraic varieties. First on computes for closed disks $\mathbf{B}_K^n$,

$$\Omega^{\mathrm{an}}_{K\langle T_i\rangle/K} = K\langle T_i\rangle \mathrm{d}T_1 + \cdots + K\langle T_i\rangle \mathrm{d}T_n.$$

Then one can extend this to define $\Omega^{\mathrm{an}}_{X/K}$ for all rigid-analytic varieties in a functorial way, meaning one can also compute the coderivative and therefore define the sheaf $\Omega^{\mathrm{an}}_{Y/X}$ via the exact sequence

$$\Omega^{\mathrm{an}}_{Y/K} \xrightarrow{\delta_f} \Omega^{\mathrm{an}}_{X/K} \longrightarrow \Omega^{\mathrm{an}}_{Y/X} \longrightarrow 0$$

In particular, this discussion implies the following comparison theorem.

Proposition A.0.3. Let $f : Y \to X$ be a morphism of schemes locally of finite type over $K$. There is a natural isomorphism

$$(\Omega_{Y/X})^{\mathrm{an}} \xrightarrow{\sim} \Omega^{\mathrm{an}}_{Y^{\mathrm{an}}/X^{\mathrm{an}}}.$$

between the analytification of the differentials and the differentials of the analytification.

Proof. The analytification of a coherent sheaf is the pullback via the morphism of locally ringed spaces $\phi \colon X^{\mathrm{an}} \to X$, and therefore the result reduces by the exact sequence above to the case of $Y = \mathbf{A}_K^n$ and $X = \operatorname{Spec} K$. Now the result follows by pulling back the isomorphism $\Omega_X \cong \mathcal{O}_X^n$. □

We now move the discussion to the cotangent complex, and its central role in deformation theory. We begin by reviewing the definition for non-topological algebras.

Definition A.0.4 (Cotangent complex). Let $R$ be a ring and $A/R$ an $A$-algebra. The (algebraic) cotangent complex of $A/R$ is defined to be

$$\mathrm{L}_{A/R} = \Omega_{P^\bullet/R} \otimes_{P^\bullet} A,$$

where $P^\bullet \to A$ is a choice of cofibrant resolution of $A$ as a simplicial $R$-module.

Such resolution always exists. For example any resolution by polynomial algebras works and the result is, of course, independent of such choices. To construct one, that is even functorial, just take $P_0 = R[A]$ and $P_{i+1} = R[P_i]$.

In practice, we can reduce the computation of $\mathrm{L}_{A/R}$ using the following principles.

Proposition A.0.5. Let $A$ be an $R$-algebra. The following statements hold.

- If $B/A$ is an $A$-algebra then we have a transitivity fiber sequence

  $$\mathrm{L}_{A/R} \otimes^L_A B \to \mathrm{L}_{B/R} \to \mathrm{L}_{A/R}$$

  in $\mathcal{D}(B)$.

- If $B = A/I$ and $A$ is a smooth $R$-algebra, we have

  $$\tau_{\leq 1}\mathrm{L}_{B/R} = \left[ I/I^2 \to \Omega_{A/R} \right].$$

  Furthermore if $I$ is defined by a regular sequence then $\mathrm{L}_{B/A} = \tau_{\leq 1}\mathrm{L}_{B/A}$.

The importance of the cotangent complex is that it controls deformations. This is an old result conjectured initially by Grothendieck and proved by Illusie on his PhD thesis. We recall here the result in the context of schemes, in the flat context, which is the one we really care about.

Theorem A.0.6 (Illusie). Let $f_0 : X_0 \to S_0$ be a flat morphism of schemes and $j : S_0 \hookrightarrow S$ be a closed immersion given by a square-zero ideal $\mathscr{I} \subset \mathscr{O}_S \twoheadrightarrow \mathscr{O}_{S_0}$. There is a obstruction class

$$o(f_0) \in \mathrm{Ext}^2_{\mathscr{O}_{X_0}}(\mathrm{L}_{X_0/S_0}, f_0^*\mathscr{I})$$

that vanishes precisely when $X_0$ admits a flat deformation $X_0 \hookrightarrow X \to S$. The isomorphism class of such solutions are a torsor under $\mathrm{Ext}^1(\mathrm{L}_{X_0/S_0}, f_0^*\mathscr{I})$, and the automorphism group of any such solution is $\mathrm{Ext}^0(\mathrm{L}_{X_0/S_0}, f_0^*\mathscr{I})$.

We now generalize the above discussion to the case relevant to us, that is, to the setting of adic spaces. Here we follow closely the exposition of [13, Sec. 7.1] (but see also [21]).

A first naive guess would be to define a (derived) $p$-complete version of the cotangent complex. Given $R_0$ a $p$-complete $\mathbf{Z}_p$-algebra, and $A \to B$ a map of $R_0$-algebras, we define

$$\widehat{\mathrm{L}}_{B/A} = \lim_n (\mathrm{L}_{A/B} \otimes^L_R \mathrm{Cof}(R \xrightarrow{p^n} R))),$$

where the limit and cofiber are taken in the $\infty$-categorical sense. This is an animated $B$-module, meaning that it lies on $\mathscr{D}_{\geqq 0}(B)$, and by analysing the $K$-flat resolution one obtains $\mathrm{H}^0(\widehat{L}_{B/A}) = \widehat{\Omega}^1_{B/A}$.

Definition A.0.7 (The analytic cotangent complex). Let $A, B$ be a pair of $p$-adic affinoid Huber pairs. The analytic cotangent complex is defined to be the filtered colimit

$$\mathrm{L}^+_{B/A} = \operatorname*{colim}_{A_0 \to B_0} \widehat{\mathrm{L}}_{B_0/A_0}, \quad \mathrm{L}^{\mathrm{an}}_{B/A} = \mathrm{L}^+_{B/A}\left[\frac{1}{p}\right]$$

taken inside the $\infty$-category $\mathscr{D}(B)^\wedge$. Here, the colimit is indexed by the filtered category of rings of definition $A_0 \subset A^+$ and $B_0 \subset B^+$.

This construction can be sheafified[6] to obtain an analytic positive cotangent complex $\mathrm{L}^+_{Y/X} \in \mathscr{D}_{\geqq 0}(\mathscr{O}^+_X)$ for analytic adic spaces $Y/X$ living over $\mathrm{Spa}\,\mathbf{Q}_p$, and finally inverting $p$ we get $\mathrm{L}^{\mathrm{an}}_{Y/X} = \mathrm{L}^+_{Y/X}[1/p] \in \mathscr{D}_{\geqq 0}(\mathscr{O}_X)$. We also have $\mathrm{H}^0(\mathrm{L}^{\mathrm{an}}_{Y/X}) = \Omega^{\mathrm{an}}_{Y/X}$.

There is a natural map $\mathrm{L}_{Y/X} \to \mathrm{L}^{\mathrm{an}}_{Y/X}$ from the topos-theoretic cotangent complex to the analytic one, which boils down to the counit $M \to \widehat{M}$ of the (derived) $p$-completion adjunction (and then inverting $p$).

Remark A.0.8. To compute this colimit, remember that the cotangent $L_{B/A}$ exists as an object of $\mathsf{Ch}_{\geqq 0}(B)$, and its terms are flat. Therefore, this can also be computed as a 1-categorical colimit in $\mathsf{Ch}_{\geqq 0}(B)$, with the transition maps being obtained by the functorial resolutions.

In particular, we deduce a functoriality with respect to maps of Huber rings, which allows us to extend the definition to adic spaces as claimed. The statement for $\mathrm{H}^0$ also follows by analysing $K$-flat resolutions.

[6] Here we are seeing $\mathrm{L}^{\mathrm{an}}_{B/A}$ as a presheaf of objects in $\mathscr{D}_{\geqq 0}$, and not in the unbounded derived category.

Remark A.0.9 ([13, Rmrk. 7.1.1, 7.1.2]). This definition can be simplified for the cases we're interested in. If $A$ is an affinoid, bounded and tft over some $p$-adic field, then

$$\operatorname*{colim}_{A^+\to B_0} \widehat{\mathrm{L}}_{B_0/A^+}\left[\frac{1}{p}\right] \xrightarrow{\sim} \mathrm{L}^{\mathrm{an}}_{A/B}$$

where now only $B_0$ varies; if $B$ is furthermore bounded, then even $\widehat{\mathrm{L}}_{B^+/A^+}[1/p] \xrightarrow{\sim} \mathrm{L}^{\mathrm{an}}_{A/B}$.

Similarly, one can also restrict only to tft rings of definition $A_0 \to B_0$ on the colimit, when $A, B$ are tft over a $p$-adic field $K$. This is due to the fact that every every ring of definition is contained in a larger tft ring of definition in this case. This recovers the definition in [21].

Proposition A.0.10. Let $X, Y$ be analytic adic spaces over $S$, with $S$ itself living over $\operatorname{Spa}\mathbf{Q}_p$, and consider an $S$-morphism $X \to Y$. Then there is an analytic fiber sequence

$$Lf^*\mathrm{L}^{\mathrm{an}}_{Y/S} \xrightarrow{\delta_f} \mathrm{L}^{\mathrm{an}}_{X/S} \longrightarrow \mathrm{L}^{\mathrm{an}}_{X/Y}$$

in $\mathscr{D}(X)$, the derived category of $\mathscr{O}_X$-modules.

Proof. All operations in the definition preserve fiber sequences. For more details, the proof of [21, Prop. 7.2.13] applies mutatis mutandis. □

From now on, we assume for simplicity that we are working over $\mathbf{B}^+_{\mathrm{dR}}/\xi^2$ (defined in next appendix). This allows us to get a proper handle on the subrings $A_0$ and $B_0$ in the definition of the analytic cotangent complex.

Proposition A.0.11. Let $X \to Y$ be a finite map of adic spaces which are tft over $\mathbf{B}^+_{\mathrm{dR}}/\xi^2$. Then the natural map

$$\mathrm{L}_{Y/X} \xrightarrow{\sim} \mathrm{L}^{\mathrm{an}}_{Y/X}$$

Is an equivalence. In particular, it follows from the properties of the usual cotangent complex that

$$\tau_{\leq 1}\mathrm{L}^{\mathrm{an}}_{X/S} \xrightarrow{\sim} \mathscr{I}/\mathscr{I}^2[1]$$

is an isomorphism, and if the immersion is regular (meaning $\mathscr{I}$ is generated by a regular sequence) then the result follows without truncation.

Proof. Follows by reducing to the affinoid case and applying [12, Prop. 5.2.15]. □

Proposition A.0.12. Let $f : Y \to X$ be a smooth map of adic spaces which are tft over $\mathbf{B}^+_{\mathrm{dR}}/\xi^2$. Then the canonical map

$$\mathrm{L}^{\mathrm{an}}_{Y/X} \xrightarrow{\sim} \Omega^{\mathrm{an}}_{Y/X}$$

is an equivalence. In particular if $Y/X$ is tft, then it is étale if and only if $\mathrm{L}^{\mathrm{an}}_{Y/X} = 0$.

Proof. This is [12, Cor. 5.2.14]. □

The following definition is not strictly necessary, but it illuminates the definition of the lift in section 3.2. We can define a deformation problem for adic spaces analogously to locally ringed spaces [21, p. 7.3.13]. An analytic deformation of a morphism of adic spaces $f: X \to S$ by a coherent $\mathscr{O}_X$-module $\mathscr{F}$ consists of a closed embedding of adic spaces $j: X \hookrightarrow Y$ together with the datum of an $\mathscr{O}_X$-linear isomorphism $j^*\mathscr{I} \xrightarrow{\sim} \mathscr{F}$, with $\mathscr{I}$ the ideal defining the embedding.

In [16, (1.4.1)] we see that in fact that any coherent ideal $\mathscr{I} \subset \mathscr{O}_Y$ which squares to zero determines an analytic extension $\mathscr{O}_X \hookrightarrow \mathscr{O}_Y$, as $\mathscr{O}_Y/\mathscr{I}$ has a canonical topology, and the stalks have canonical valuations, allowing us to define an adic space.

We denote by $\mathrm{Exan}_S(X, \mathscr{F})$ the category of such deformations. A morphism of extensions is a map of adic spaces $Y \to Y'$ over $X$ which makes the diagram

$$\begin{array}{ccccccccc} 0 & \longrightarrow & \mathscr{F} & \longrightarrow & j^*\mathscr{O}_{Y'} & \longrightarrow & \mathscr{O}_X & \longrightarrow & 0 \\ & & \| & & \downarrow\wr & & \| & & \\ 0 & \longrightarrow & \mathscr{F} & \longrightarrow & j^*\mathscr{O}_Y & \longrightarrow & \mathscr{O}_X & \longrightarrow & 0 \end{array}$$

commute. We note that $\mathrm{Exan}_S(X, \mathscr{F})$ is a groupoid, and even a Picard groupoid via the usual arguments with Baer sums.

# B Sites associated to rigid-spaces

We recall that a perfectoid space is a certain type of adic space which is glued locally from perfectoid Tate pairs. We denote by Perfd (resp. Perf) the full subcategory of adic spaces spanned by perfectoid spaces (resp. perfetoid spaces of characteristic $p$). Perfectoid spaces admit a good theory of étale morphisms and also pro-étale morphisms as recalled below.

Definition B.0.1. Let $X = \mathrm{Spa}(R, R^+)$ and $Y = \mathrm{Spa}(S, S^+)$ be affinoid perfectoids. A morphism $f: Y \to X$ is said to be affinoid pro-étale if $Y$ can be written as the limit $Y = \lim Y_i \to X$ where $Y_i = \mathrm{Spa}(S_i, S_i^+) \to X$ are étale affinoids. That is, for a pseudo-uniformizer $\varpi$ of $X$ we have that $S$ is isomorphic to the $\varpi$-adic completion

$$S^+ = (\mathrm{colim}\, S_i^+)^\wedge_\varpi, \qquad S = S^+\left[\frac{1}{\varpi}\right]$$

of the colimit with $\mathrm{Spa}(S_i, S_i^+) \to X$ étale.

A morphism of perfectoid spaces is said to be pro-étale if it is locally on source and target affinoid proétale.

Remark B.0.2. In general, one might try and define pro-étale morphisms for general adic spaces using the pro-category of $X_{\text{ét}}$, as in [22], which works well for perfectoids and locally Noetherian adic spaces, and leads to what is called the flattened pro-étale site of a rigid space. However, some care with covers might be needed to define a pro-étale topology [23], and so we've opted to follows the more modern approach of [24] using the related notion of a quasi-pro-étale morphisms (see below).

Pro-étale morphism have the expected permanence properties. They are closed under composition, base-change, and any morphism between pro-étale morphisms is pro-étale. However, they do not satisfy pro-étale descent [25, Example 9.1.15].

Definition B.0.3. Let Perf be the category of characteristic $p$ perfectoid spaces.

A collection of morphisms $\{Y_i \to X\}$ of perfectoid spaces is a v-cover if for all quasi-compact opens $U$ of $X$ there is a finite subset $Y_{i_1}, \ldots, Y_{i_n}$ of the $Y_i$ and quasi-compact opens $V_i \subset Y_i$ such that

$$\bigcup_{d=0}^{n} f(V_{i_d}) = U.$$

The v-topology is the Grothendieck topology on Perf which is generated by v-covers.

A v-cover $\{Y_i \to X\}$ is said to be a pro-étale cover if all maps $Y_i \to X$ are pro-étale. The pro-étale topology on Perf is the Grothenidieck topology generated by such covers.

Rigid-analytic spaces can be reincarnated as certain pro-étale sheaves as we will see below. But first, we describe a local version of pro-étale maps which can be extended to sheaves on $\mathrm{Perf}_{\texttt{proét}}$.

Definition B.0.4 (Quasi-pro-étale maps). A perfectoid space $X$ is said to be strictly totally disconnected, if it is qcqs and every étale cover of $X$ splits. Equivalently [24, Prop. 7.16], every connected component of $X$ is of the form $\mathrm{Spa}(C, C^+)$ for $C$ an algebraically closed perfectoid field.

A morphism of pro-étale stacks $f : Y \to X$ is said to be quasi-pro-étale if it is locally separated (meaning separated locally in the domain), and for all strictly totally disconnected $X'$ and maps $X' \to X$ the pullback $Y_{X'} \to X'$ is pro-étale.

By taking a careful limit over enough affinoid open covers of some space, we see that every perfectoid space is pro-étale locally strictly totally disconnected [24, Lemma 7.18]. In particular quasi-pro-étale morphisms are pro-étale locally pro-étale. We also see that morphisms which are (quasi-)pro-étale locally quasi-pro-étale are quasi-pro-étale.

Definition B.0.5 ([24, Def. 11.1]). A diamond $X$ is a pro-étale sheaf on the site Perf of characteristic $p$ perfectoid spaces, which is the quotient of a perfectoid $Y$ by a pro-étale equivalence relation, that is, a relation $R \subset Y \times Y$ such that the projection maps $R \rightrightarrows Y$ are pro-étale.

Equivalently, a pro-étale sheaf $X$ is a diamond if it admits a quasi-pro-étale surjection from a perfectoid (cf. Prop. 11.5).

Therefore diamonds are analogous to algebraic spaces, however we do not ask for the representability of the diagonal because this requirement is too strong in this setting. There is a good notion of analytic topology associated to a diamond $X$, namely we define the topological space

$$|X| = |Y|/|R|,$$

where $Y \to X$ is a quotient by a pro-étale relation $R = Y \times_X Y$.

Definition B.0.6. Let $X$ be a locally spatial diamond, then we define the following sites in increasing order of fineness.

- The analytic site $X_{\text{an}}$ which is the site associated to the topological space $|X|$.
- The (finite) étale site $X_{\text{ét}}$ ($X_{\text{fét}}$), whose objects are (finite) étale morphisms $Y \to X$ and v-covers.
- The quasi-pro-étale site $X_{\text{qproét}}$, whose objects are quasi-pro-étale morphisms $Y \to X$ and same covers.
- The v-site $X_{\text{v}}$, whose objects are morphisms from a spatial diamond $Y \to X$ and same covers.

Remark B.0.7. Some care must be taken to circumvent set-theoretic issues for the qproét and v sites. For this purpose, we fix a cutoff cardinal $\kappa$ and take a colimit (loc. cit. Sec. 4). This procedure is shown to preserve cohomology, and therefore we will not need to mention $\kappa$ explicitly in any of the following results.

Remark B.0.8. We mention the v-site mostly for completeness, as other references work with v-bundles. However, in view of theorem 1.1.1, we can work with the quasi-pro-étale site instead.

We note that, by design, the quasi-pro-étale topos of a diamond is locally perfectoid. Indeed, any diamond $X$ is covered by a perfectoid, and hence we can pullback this cover to any $Y \in X_{\text{qproét}}$. In particular, since perfectoids are locally weakly contractible [8, Def. 3.2.1], the quasi-pro-étale topos of $X$ is locally weakly contractible and hence replete by [8, Prop. 3.2.3]. This will be important for us in the sequel.

There are natural canonical maps of ringed sites

$$X_{\text{v}} \xrightarrow{\lambda} X_{\text{qproét}} \xrightarrow{\nu} X_{\text{ét}}$$

given by the inclusion functors. These functors will turn out to be an essential part of the correspondence. We note that the pullback via $\lambda$ and $\nu$ are fully faithful, and the cohomology of étale sheaves agree on all three sites. [24, Props. 14.7, 14.8]

Definition B.0.9 ([24, Def. 15.5]). Let $X$ be an analytic adic space over $\mathbf{Z}_p$. We define the diamond associated to $X$ to be the pro-étale sheaf

$$X^{\diamond}\colon \mathsf{Perf} \to \mathsf{Set} \qquad S \mapsto \left\{(S^{\sharp},\iota),\ f : S^{\sharp} \to X\right\}/\cong,$$

where $S^{\sharp}$ is a perfectoid space, $\iota\colon (S^{\sharp})^{\flat} \xrightarrow{\sim} S$ is an isomorphism, and $f$ is a morphism of adic spaces. This data is considered up to isomorphism of such triples.

Theorem B.0.10 ([24, Lemma. 15.6]). Let $X$ be an analytic adic space over $\operatorname{Spa}\mathbf{Z}_p$. We have $|X^{\diamond}| = |X|$, and $X^{\diamond}$ is locally spatial (and therefore spatial if $X$ is qcqs). Furthermore, $X^{\diamond}$ is a diamond, and hence it is qproét and v locally perfectoid.

The associated diamond functor preserves étaleness, and induces an equivalence of (finite) étale sites $X_{\text{ét}} \cong X^{\diamond}_{\text{ét}}$, $X_{\text{fét}} \cong X^{\diamond}_{\text{fét}}$.

We note that when proving that when proving that $X^\diamond$ there are two keys steps: the first is to show that this is indeed a pro-étale sheaf, which reduces to showing that the functor

$$(\mathrm{Spa}\,\mathbf{Z}_p)^\diamond \colon S \mapsto \left\{(S^\sharp, \iota) \mid \iota(S^\sharp)^\flat \xrightarrow{\sim} S\right\}/\cong$$

parametrizing untilts of $S$ is a pro-étale sheaf. The next step is to show that any analytic adic space over $\mathrm{Spa}\,\mathbf{Z}_p$ is the quotient of a perfectoid by a pro-étale equivalence relation. This information is crucial to the proof of our main theorem.

We also note that the procedure $X \mapsto X^\diamond$ does lose some information, as this functor is not fully faithful. This procedure preserves information of a more "topological" nature (such as the étale site).

Now if $X$ is a rigid analytic variety, then we have a structure sheaf $\mathcal{O}_X$ on $X_{\text{ét}}$, which we can pullback to a quasi-pro-étale/v-sheaf. We can then define the following completed version of the structure sheaf on these sites.

Definition B.0.11 (The completed structure sheaf). Let $X$ be an analytic adic space over a non-archemidean field $K$. The completed structure sheaf, or the quasi-pro-étale structure sheaf, $\widehat{\mathcal{O}}_X$ is defined to be the sheaf

$$\widehat{\mathcal{O}}_X^+ = \lim_n \nu^{-1}\mathcal{O}_X^+/p^n; \quad \widehat{\mathcal{O}}_X = \widehat{\mathcal{O}}_X^+\left[\frac{1}{p}\right]$$

where the limit is taken as quasi-pro-étale sheaves. A similar definition also works within $X_{\mathrm{v}}$.

The cohomology of $\widehat{\mathcal{O}}_X$ captures interesting phenomena of the quasi-pro-étale topology of rigid-analytic varieties. First, there is a acyclicity phenomena for affinoid perfectoid.

Theorem B.0.12 ([24, Prop. 8.5 (iii)]). Let $X$ be a rigid-analytic space over $K$ and $Y = \mathrm{Spa}(R, R^+) \in X_{\mathtt{qproét}}$ an affinoid perfectoid. Then $\mathcal{O}_X(Y) \xrightarrow{\sim} \widehat{\mathcal{O}}_X(Y)$ and $R\Gamma(Y, \widehat{\mathcal{O}}_X) = \Gamma(Y, \mathcal{O}_X) = R$.

In particular, if $X$ is a rigid-analytic variety, which from now on we always see as a diamond, and $Y \to X$ is an affinoid perfectoid quasi-pro-étale cover of $X$, then we can compute the cohomology $R\Gamma(X, \widehat{\mathcal{O}}_X)$ as the Čech nerve of this cover.

On the other hand, if $X$ is proper, then the cohomology of $\widehat{\mathcal{O}}_X$ actually captures the étale cohomology of $X$. First, note that We can consider $C$ as a sheaf of rings on the $\mathtt{qproét}$-site of $X$ by considering continuous maps into it; that is, we consider $C$ as the sheaf

$$\underline{C} \colon X_{\mathtt{qproét}}^{\mathrm{op}} \to \mathsf{Rings}, \quad Y \mapsto \mathrm{Hom}(|Y|, C),$$

where the morphisms are taken in the category of topological spaces. When it is clear from context, we will denote the sheaf $\underline{C}$ by simply $C$. We note that by repleteness of the quasi-pro-étale topos, we have that $\underline{\mathcal{O}_C} = R\lim_n \underline{\mathcal{O}_C}/p^n$ [8, Prop. 3.1.10], and hence $R\Gamma(X, C)$ computes the étale cohomology of $X$ with $C$-coefficients.

Theorem B.0.13 (The generic version of the primitive comparision theorem). Let $X$ be a proper rigid analytic variety over $C$. Then the natural map $C \to \widehat{\mathcal{O}}_X$ induces an equivalence of $\mathbf{E}_\infty$-algebras

$$R\Gamma(X, C) \xrightarrow{\sim} R\Gamma(X, \widehat{\mathcal{O}}_X),$$

where we are taking global sections in qproét site [7].

Proof. The primitive comparison theorem [18, Cor. 3.9.24], says that

$$R\Gamma(X, \mathscr{O}_C/p) \xrightarrow[a]{\sim} R\Gamma(X, \widehat{\mathscr{O}}_X^+/p)$$

is an almost quasi-isomorphism. The result now follows from a "almost derived Nakayama" argument: from the exact sequences $0 \to p^n\mathscr{O}_C/p^{n+1} \to \mathscr{O}_C/p^{n+1} \to \mathscr{O}_C/p^{n+1} \to 0$ and $0 \to p^n\widehat{\mathscr{O}}_X^+/p^{n+1} \to \widehat{\mathscr{O}}_X^+/p^{n+1} \to \widehat{\mathscr{O}}_X^+/p^n \to 0$ we get by induction

$$R\Gamma(X, \mathscr{O}_C/p^n) \xrightarrow[a]{\sim} R\Gamma(X, \widehat{\mathscr{O}}_X^+/p^n),$$

so now the result follows by taking $R\lim$, since $R\Gamma$ commutes with it and by the fact that $X_{\text{qproét}}$ is replete we have $\mathscr{O}_C = R\lim_n \mathscr{O}_C/p^n$ and $\widehat{\mathscr{O}}_X^+ = R\lim_n \widehat{\mathscr{O}}_X^+/p^n$. □

Remark B.0.14. This proof, as stated, follows from the development of a 6-functor formalism of $p$-torsion sheaves on diamonds. The original statement is due to Scholze on [22, Thm. 1.3], using the flattened pro-étale topology.

# References


[1] Ahmed Abbes, Michel Gros, and Takeshi Tsuji. The p-adic Simpson Correspondence (AM-193). Vol. 193. Princeton University Press, 2016.

[2] Grigory Andreychev. "Pseudocoherent and Perfect Complexes and Vector Bundles on Analytic Adic Spaces". In: (2021). DOI: 10.48550/ARXIV.2105.12591. URL: https://arxiv.org/abs/2105.12591.

[3] Johannes Anschütz and Arthur-César Le Bras. "A Fourier Transform for Banach-Colmez spaces". In: (2021). DOI: 10.48550/ARXIV.2111.11116. URL: https://arxiv.org/abs/2111.11116.

[4] Johannes Anschütz, Ben Heuer, and Arthur-César Le Bras. "Hodge-Tate stacks and non-abelian $p$-adic Hodge theory of v-perfect complexes on smooth rigid spaces". In: (2023). DOI: 10.48550/ARXIV.2302.12747. URL: https://arxiv.org/abs/2302.12747.

[5] Johannes Anschütz, Ben Heuer, and Arthur-César Le Bras. The small $p$-adic Simpson correspondence in terms of moduli spaces. 2023. arXiv: 2312.07554 [math.AG].

[6] Bhargav Bhatt. "The Hodge-Tate decomposition via perfectoid spaces Arizona Winter School 2017". In: (2017).

[7] Bhargav Bhatt, Matthew Morrow, and Peter Scholze. "Integral p-adic Hodge theory". In: (2016). DOI: 10.48550/ARXIV.1602.03148. URL: https://arxiv.org/abs/1602.03148.


[7] Here we can also use the v-site using an analogous structure sheaf $\check{\mathscr{O}}_X$ (see [19, Def. 2.1]). The relevance of this remark is that the proof technically involves v-cohomology, but this is inessential [19, Lemma 2.9].

[8] Bhargav Bhatt and Peter Scholze. "The pro-étale topology for schemes". In: (2013). DOI: 10.48550/ARXIV.1309.1198. URL: https://arxiv.org/abs/1309.1198.

[9] P. Deligne. "Le groupe fondamental de la droite projective moins trois points". In: (1989), pp. 79–297.

[10] Christopher Deninger and Annette Werner. "Vector bundles on p-adic curves and parallel transport". In: 38.4 (2005), pp. 553–597.

[11] Gerd Faltings. "A p-adic Simpson correspondence". In: Advances in Mathematics 198.2 (2005), pp. 847–862.

[12] Haoyang Guo. "Crystalline cohomology of rigid analytic spaces". In: (2021). DOI: 10.48550/ARXIV.2112.14304. URL: https://arxiv.org/abs/2112.14304.

[13] Haoyang Guo. "Hodge-Tate decomposition for non-smooth spaces". In: (2019). DOI: 10.48550/ARXIV.1909.09917. URL: https://arxiv.org/abs/1909.09917.

[14] Ben Heuer. "A $p$-adic Simpson correspondence for smooth proper rigid varieties". In: (2023). arXiv: 2307.01303 [math.AG].

[15] Ben Heuer. "Line bundles on rigid spaces in the $v$-topology". In: (2020). DOI: 10.48550/ARXIV.2012.07918. URL: https://arxiv.org/abs/2012.07918.

[16] Roland Huber. Étale cohomology of rigid analytic varieties and adic spaces. Vol. 30. Springer, 2013.

[17] Jacob Lurie. "Higher algebra". In: Preprint (2016). URL: https://people.math.harvard.edu/~lurie/papers/HA.pdf.

[18] Lucas Mann. "A $p$-Adic 6-Functor Formalism in Rigid-Analytic Geometry". In: arXiv preprint arXiv:2206.02022 (2022).

[19] Lucas Mann and Annette Werner. "Local systems on diamonds and $p$-adic vector bundles". In: (2020). DOI: 10.48550/ARXIV.2005.06855. URL: https://arxiv.org/abs/2005.06855.

[20] Madhav V Nori. "The fundamental group-scheme". In: Proceedings Mathematical Sciences 91.2 (1982), pp. 73–122.

[21] Lorenzo Ramero Ofer Gabber. Almost Ring Theory. 1st ed. Lecture Notes in Mathematics 1800. Springer, 2003. ISBN: 3540405941; 9783540405948; 9783540450962; 3540450963. URL: libgen.li/file.php?md5=8b69b48131562d57909b60c9309e2590.

[22] Peter Scholze. "$p$-adic Hodge theory for rigid-analytic varieties". In: Forum of Mathematics, Pi 1 (2013).

[23] Peter Scholze. "Erratum to '$p$-adic Hodge theory for rigid-analytic varieties"'. In: ().

[24] Peter Scholze. "Etale cohomology of diamonds". In: (2017). DOI: 10.48550/ARXIV.1709.07343. URL: https://arxiv.org/abs/1709.07343.

[25] Peter Scholze and Jared Weinstein. Berkeley Lectures on p-adic Geometry:(AMS-207). Princeton University Press, 2020.

[26] Yupeng Wang. "A $p$-adic Simpson correspondence for rigid analytic varieties". In: Algebra Number Theory 17.8 (2023), pp. 1453–1499. ISSN: 1937-0652,1944-7833. DOI: 10.2140/ant.2023.17.1453. URL: https://doi.org/10.2140/ant.2023.17.1453.